I. WIGNER FUNCTION FOR ANHARMONIC OSCILLATOR

We consider Hamiltonian of a harmonic oscillator perturbed by a polynomial potential,

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(p_i^2 + \omega_i^2 q_i^2\right) + \lambda \sum_{\{k_i\}} f_{\{k_i\}} q_1^{k_1} q_2^{k_2} q_1^{k_1} ... q_N^{k_2}, \tag{1}$$

where summation is performed over N-component multi-indexes $\{k_i\}$. Here, we restrict ourselves to cubic anharmonic terms with $k_1 + k_2 + ... + k_N = 3$, although this method could be extended in principle to higher anharmonic corrections. In (1), λ is a small perturbation parameter.

The perturbed Wigner-function is expanded in powers of λ,

$$\rho(\vec{p}, \vec{q}) = \rho_0(\vec{p}, \vec{q}) + \lambda \rho_1(\vec{p}, \vec{q}) + O(\lambda^2), \tag{2}$$

where

$$\rho_0(\vec{p}, \vec{q}) = \exp\left(-\sum_{i=1}^{N}\left(\frac{p_i^2}{\omega_i} + \omega_i q_i^2\right)\right) \tag{3}$$

is the harmonic-oscillator Wigner function, and $\rho_1(\vec{p}, \vec{q})$ is the first anharmonic correction to be determined here.

To develop perturbation theory for the ground state wave function $\Psi(\vec{q})$, it is convenient to rewrite the Scrodinger equation in terms of the function $S(\vec{q}) = -\ln \Psi(\vec{q})$ []:

$$-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial S}{\partial q_i}\right)^2 + \frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2 S}{\partial q_i^2} + V(\vec{q}) - E = 0 \tag{4}$$

Without the second sum, Eq. (4) reduces to Hamilton - Jacobi equation for action of a classical particle moving in a potential $E - V(\vec{q})$. In such a quasiclassical limit, perturbation theory for $S(\vec{q})$ is easily developed []. More general quantum case which is considered here is still solvable analytically, but resulting corrections have more monomial terms.

Let us expand quantities entering Eq. (4) in powers of λ:

$$S(\vec{q}) = S_0(\vec{q}) + \lambda S_1(\vec{q}) + O(\lambda^2), \tag{5}$$

$$V(\vec{q}) = V_0(\vec{q}) + \lambda V_1(\vec{q}) + O(\lambda^2), \tag{6}$$

$$E = E_0 + \lambda E_1 + O(\lambda^2). \tag{7}$$

Zero-order terms are:

$$S_0(\vec{q}) = \frac{1}{2}\sum_{i=1}^{N}\omega_i q_i^2 \tag{8}$$

$$V_0(\vec{q}) = \frac{1}{2}\sum_{i=1}^{N}\omega_i^2 q_i^2 \tag{9}$$

$$E_0 = \frac{1}{2}\sum_{i=1}^{N}\omega_i. \tag{10}$$

First-order correction to the potential, see Eq. (1), is

$$V_1(\vec{q}) = \sum_{k_1+k_2+...+k_N=3} f_{\{k_i\}} q_1^{k_1} q_2^{k_2} q_1^{k_1} ... q_N^{k_2}. \tag{11}$$

Expansions (5) are substituted into Eq. (4), and the left hand side is expanded in λ. In the first order in λ, Eq. (4) is equivalent to

$$-\sum_{i=1}^{N}\frac{\partial S_0}{\partial q_i}\frac{\partial S_1}{\partial q_i} + \frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2 S_1}{\partial q_i^2} + V_1(\vec{q}) - E_1 = 0 \tag{12}$$

which is a linear equation in respect to a number E_1 and a function $S_1(\vec{q})$.

Let us suppose that $S_1(\vec{q})$ is a polynomial

$$S_1(\vec{q}) = \sum_{\{k_i\}} s_{\{k_i\}} q_1^{k_1} q_2^{k_2} q_1^{k_1} ... q_N^{k_2}. \tag{13}$$

Then Eq. (12) is equivalent to a set of linear equations in respect to coefficients $s_{\{k_i\}}$ and E_1 ,

$$\sum_{i=1}^{N}\left[-k_i\omega_i s_{\{k_i\}} + \frac{1}{2}(k_i+1)(k_i+2)s_{\{k_1,k_2,...,k_i+2,...,k_N\}}\right] + f_{\{k_i\}} - \delta\left(\{k_i\},\{0\}\right)E_1 = 0, \tag{14}$$

where coefficients are supposed to be zero if one of indexes is negative.

Since inhomogeneous part of Eq. (12) , $V_1(\vec{q}) - E_1$, is a third degree polynomial, it may be shown that the solution is at most a third degree polynomial too,

$$S_1(\vec{q}) = \sum_{k_1+k_2+...+k_N=1} A_{\{k_i\}} q_1^{k_1} q_2^{k_2} q_1^{k_1} ... q_N^{k_2} + \sum_{k_1+k_2+...+k_N=2} B_{\{k_i\}} q_1^{k_1} q_2^{k_2} q_1^{k_1} ... q_N^{k_2}$$
$$+ \sum_{k_1+k_2+...+k_N=3} C_{\{k_i\}} q_1^{k_1} q_2^{k_2} q_1^{k_1} ... q_N^{k_2}. \tag{15}$$

In (15) , we split (13) in three homogeneous polynomials of the first, the second, and the third degrees. Now, recurrence relation (14) between polynomial coefficients reads

$$B_{\{2,0,0,\dots,0\}} + B_{\{0,2,0,\dots,0\}} + \dots + B_{\{0,0,0,\dots,2\}} - E_1 = 0 \tag{16}$$

$$\sum_{i=1}^{N}[-k_i\omega_i A_{\{k_i\}} + \frac{1}{2}(k_i+1)(k_i+2)C_{\{k_1,k_2,\dots,k_i+2,\dots,k_N\}}] = 0 \tag{17}$$

$$\sum_{i=1}^{N} k_i\omega_i B_{\{k_i\}} = 0 \tag{18}$$

$$-\sum_{i=1}^{N} k_i\omega_i C_{\{k_i\}} + f_{\{k_i\}} = 0 \tag{19}$$

It follows from (19) that

$$C_{\{k_i\}} = \left(\sum_{i=1}^{N} k_i\omega_i\right)^{-1} f_{\{k_i\}}. \tag{20}$$

It follows from (18) that $B_{\{k_i\}} = 0$, and from (16) that $E_1 = 0$. Finally, from (17)

$$A_{\{k_i\}} = \frac{1}{2}\left(\sum_{i=1}^{N} k_i\omega_i\right)^{-1} \sum_{i=1}^{N}(k_i+1)(k_i+2)C_{\{k_1,k_2,\dots,k_i+2,\dots,k_N\}}. \tag{21}$$

Since the perturbed wavefunction is $\Psi(\vec{q}) = [1 - \lambda S_1(\vec{q})]\exp(-S_0(\vec{q})) + O(\lambda^2)$, then $\rho_1(\vec{p},\vec{q})$ is a contribution to Wigner transform from integration of the function, $-2S_1(\vec{q} + \vec{\eta}/2)\exp(-S_0(\vec{q}+\vec{\eta}/2))\exp(-S_0(\vec{q}-\vec{\eta}/2))$. Substituting there expansion (15), we finally find

$$\begin{aligned}\rho_1(\vec{p},\vec{q}) = &-\sum_{k_1+k_2+\dots+k_N=1} A_{\{k_i\}}F_{k_1,\omega_1}(p_1,q_1)F_{k_2,\omega_2}(p_2,q_2)\dots F_{k_N,\omega_N}(p_N,q_N)\\ &-\sum_{k_1+k_2+\dots+k_N=3} C_{\{k_i\}}F_{k_1,\omega_1}(p_1,q_1)F_{k_2,\omega_2}(p_2,q_2)\dots F_{k_N,\omega_N}(p_N,q_N).\end{aligned} \tag{22}$$

where

$$F_{k,\omega}(p,q) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\eta(q+\eta/2)^k \exp\left(-\frac{\omega}{2}(q+\eta/2)^2 - \frac{\omega}{2}(q-\eta/2)^2\right)\cos(p\eta). \tag{23}$$

It can be expressed analitically as

$$F_{0,\omega}(p,q) = \frac{2}{\sqrt{\pi\omega}}\exp\left(-\frac{p^2}{\omega} - \omega q^2\right),$$
$$F_{1,\omega}(p,q) = \frac{2q}{\sqrt{\pi\omega}}\exp\left(-\frac{p^2}{\omega} - \omega q^2\right),$$

$$F_{2,\omega}(p,q) = \frac{\omega - 2p^2 + 2\omega^2 q^2}{\sqrt{\pi\omega}\omega^2} \exp\left(-\frac{p^2}{\omega} - \omega q^2\right),$$
$$F_{3,\omega}(p,q) = \frac{q(3\omega - 6p^2 + 2\omega^2 q^2)}{\sqrt{\pi\omega}\omega^2} \exp\left(-\frac{p^2}{\omega} - \omega q^2\right). \tag{24}$$

Indexes higher than 3 don't enter (22).

Final result of this section is given by Eq. (22), (24), (20), and (21).