Dependence of the singularity of Borel function on coupling parameter 1/Z for helium Hamiltonians

We shall use the following one-dimensional prototype Hamiltonian modelling inter-particle Coulomb interaction:

$$H = \delta^2 \frac{p^2}{2} + \frac{x^2}{2} + \frac{g}{x-1} .$$

The term $\delta^2 \frac{p^2}{2} + \frac{x^2}{2}$ models a harmonic oscillator potential around the equilibrium point in an effective potential (including the centrifugal term), and the term $\frac{g}{x-1}$ models the interaction with a Coulombic center at the point $x = 1$.

Here, we are looking for large-order asymptotics of the coefficients of δ-expansion for the energy,

$$E = \sum_{k=0}^{\infty} E_k \delta^k ,$$
$$E_k \sim C k^{\beta} a^k k!, \quad k \to \infty .$$

The parameter $\delta_0 = 1/a$ is known as a singularity of the Borel function for the energy. Our aim is to prove that δ_0 tends to a finite value in the small-coupling limit $g \to 0$. We believe that the same qualitative behavior of the Borel singularity takes place for the realistic two-particle helium Hamiltonian. Taking advantage of the simplicity of our model, we calculate δ_0 exactly (in analytic form) as a function of g. Then, we investigate the behavior of this function at small g and prove that it has a finite limit at $g \to 0$. Finally, in order to back up our conjecture, we calculate numerically the Borel singularities for helium Hamiltonians as a function of $g = 1/Z$ and confirm that they have the same qualitative behavior at $g \to 0$ as our simple model.