Diamagnetic hydrogen atom at the limit of large field and large dimensionality (or magnetic quantum number *m*)

Here we suppose that a magnetic field B and a magnetic quantum number m both tend to infinity while their ratio tends to a constant. We introduce a small parameter $\delta = \frac{1}{m+a}$ and perform the expansion of the energy in powers of δ assuming that $\tilde{B} = \delta B$ equals to a constant. If we choose the shift parameter $a = 1/2$ then the quantity $\tilde{B}$ coincides with p from a paper of W. Rosner et al. (1983).

In cylindrical coordinates, the Schrödinger equation takes the form:

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{m^2-1/4}{2\rho^2}-\frac{1}{r}+\frac{B^2}{8}\rho^2-E\right]\psi(\rho,z)=0\quad, \tag{1}$$

where $r=(\rho^2+z^2)^{1/2}$, and the wave function $\Psi(\mathrm{r})=e^{im\varphi}\rho^{-1/2}\psi(\rho,z)$. Multiplying (1) by δ^2 we arrive to the equation

$$\left[-\frac{\delta^2}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1-2a\delta+(a^2-1/4)\delta^2}{2\rho^2}-\frac{\delta^2}{r}+\frac{\tilde{B}^2}{8}\rho^2-\tilde{E}\right]\psi(\rho,z)=0\,, \tag{2}$$

where $\tilde{E}=\delta^2 E$ is scaled energy. In a classical limit $\delta\to 0$, the wavefunction $\psi(\rho,z)$ concentrates around the minimum of the potential $\frac{1}{2\rho^2}+\frac{\tilde{B}^2}{8}\rho^2$ on the line $\rho=\rho_0$, $\rho_0=(2/\tilde{B})^{1/2}$, and the energy tends to a scaled Landau energy $\tilde{E}^{(0)}=\tilde{B}/2$.

To calculate corrections to the classical limit, let us introduce displacement coordinate $x=(\rho-\rho_0)\delta^{1/2}$. Using the expansion $r^{-1}=[(\rho_0+\delta^{1/2}x)^2+z^2]^{-1/2}=(\rho_0+z^2)^{-1/2}+O(\delta^{1/2})$ and keeping terms up to order $\delta^{1/2}$ we obtain the uncoupled equations

$$\begin{aligned}&\left[-\frac{\delta}{2}\frac{d^2}{dx^2}+\frac{1-2a\delta+(a^2-1/4)\delta^2}{2(\rho_0+\delta^{1/2}x)^2}+\frac{\tilde{B}^2}{8}(\rho_0+\delta^{1/2}x)^2-\tilde{E}^{(1)}\right]\psi^{(1)}(x)=0\\&\left[-\frac{\delta^2}{2}\frac{d^2}{dz^2}-\delta^2(\rho_0^2+z^2)^{-1/2}-\tilde{E}^{(2)}\right]\psi^{(2)}(x)=0\end{aligned}\quad, \tag{3}$$

The eigenvalue of the first equation in (3) equals to the scaled Landau energy, $\tilde{E}^{(1)} = \delta^2 E_{\mathrm{Landau}} = \delta^2 \frac{B}{2}(m + 2n_1 + 1) = \delta^2 \frac{B}{2}(\delta^{-1} - a + 2n_1 + 1) = \frac{\tilde{B}}{2} + (2n_1 + 1 - a)\frac{\tilde{B}}{2}\delta$, and the eigenvalue of the second equation in (3) equals to the scaled binding energy $\tilde{E}_{\mathrm{B}} = \delta^2 E_{\mathrm{B}} = \delta^2 \varepsilon^{(2)}$ where $\varepsilon^{(2)}$ is the eigenvalue in the equation

$$\left[-\frac{1}{2}\frac{d^2}{dz^2} - (\rho_0^2 + z^2)^{-1/2} - \varepsilon^{(2)}\right]\psi^{(2)}(x) = 0\,, \tag{4}$$

Note that if the shift parameter *a* is chosen to be 1/2, then the potential in Eq. (4) is identical to „asymptotic“ potential of W. Rosner et al. (1983), and the binding energy equals to their „asymptotic“ energy.