Diamagnetic hydrogen atom at the limit of large field and large dimensionality (or magnetic quantum number *m*)

Here we suppose that a magnetic field B and a magnetic quantum number m both tend to infinity while their ratio tends to a constant. We introduce a small parameter $\delta = \frac{1}{m+a}$ and perform the expansion of the energy in powers of δ assuming that $\tilde{B} = \delta B$ equals to a constant. If we choose the shift parameter $a = 1/2$ then the quantity $\tilde{B}$ coincides with p from a paper of W. Rosner et al. (1983).

In cylindrical coordinates, the Schrödinger equation takes the form:

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{m^2-1/4}{2\rho^2}-\frac{1}{r}+\frac{B^2}{8}\rho^2-E\right]\psi(\rho,z)=0 \quad , \qquad (1)$$

where $r=(\rho^2+z^2)^{1/2}$, and the wave function $\Psi(\mathrm{r})=e^{im\varphi}\rho^{-1/2}\psi(\rho,z)$. Multiplying (1) by δ^2 we arrive to the equation

$$\left[-\frac{\delta^2}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1-2a\delta+(a^2-1/4)\delta^2}{2\rho^2}-\frac{\delta^2}{r}+\frac{\tilde{B}^2}{8}\rho^2-\tilde{E}\right]\psi(\rho,z)=0\,, \qquad (2)$$

where $\tilde{E}=\delta^2 E$ is scaled energy. In a classical limit $\delta\to 0$, the wavefunction $\psi(\rho,z)$ concentrates around the minimum of the potential $\frac{1}{2\rho^2}+\frac{\tilde{B}^2}{8}\rho^2$ on the line $\rho=\rho_0$, $\rho_0=(2/\tilde{B})^{1/2}$, and the energy tends to a scaled Landau energy $\tilde{E}^{(0)}=\tilde{B}/2$.

To calculate corrections to the classical limit, let us introduce displacement coordinate $x=m^{1/2}(\rho-\rho_0)$ and rewrite eq. (2):

$$\left[\begin{array}{l}-\frac{1}{2}\frac{\partial^2}{\partial x^2}-\frac{\delta}{2}\frac{\partial^2}{\partial z^2}+\frac{\tilde{B}^2}{2}x^2-2^{-3/2}\tilde{B}^{5/2}x^3\delta^{1/2}+\frac{5}{16}\tilde{B}^3x^4\delta-\frac{\tilde{B}}{2}a+\ldots\\ -(\rho_0^2+z^2)^{-1/2}\delta+\rho_0 x(\rho_0^2+z^2)^{-3/2}\delta^{3/2}+\ldots-\bar{E}\end{array}\right]\varphi(x,z)=0\,, \qquad (3)$$

where $\bar{E}=(\tilde{E}-\tilde{B}/2)\delta^{-1}$. In the lowest order in δ, the wavefunction is separable:

$$\varphi(x,z) = \varphi_{n_1}(x) f_{n_2}(z)\,, \tag{4}$$

where $\varphi_{n_1}(x)$ is a harmonic oscillator eigenfunction, and $f_{n_2}(z)$ is an eigenfunction in the potential $\left(\rho_0^2 + z^2\right)^{-1/2}$. Up to the first order in δ, the energy in eq. (3) is:

$$\overline{E} = (n_1 + 1/2 - a)\tilde{B} + \ldots + \varepsilon_{n_2}\delta\,, \tag{5}$$

where ε_{n_2} is the eigenvalue corresponding to $f_{n_2}(z)$.

Within the approximation (5), the binding energy $E_{\mathrm{B}} = E_{\mathrm{Landau}} - E$ equals to $\varepsilon_{n_2}\delta$, where $E_{\mathrm{Landau}} = (m + 2n_1 + 1)\dfrac{B}{2}$. So, our result coincides with „asymptotic“ energy introduced by W. Rosner et al. (1983).