The 1/n-expansion in quantum mechanics and quasi-stationary states

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Abstract. Applications of the 1/n-expansion to energy and wavefunction calculations, especially for quasi-stationary states, are considered. As an illustration we have examined the funnel and Yukawa potentials, the Stark effect in a strong field and a hydrogen atom in electric and magnetic fields. We have found that in many cases the method ensures high accuracy, even for small quantum number, $n \sim 1$. The connection between these results and the properties of coherent states is briefly discussed. Finally, the behaviour of the large-order terms in the 1/n-expansion is investigated.

1. Introduction

The 1/N-expansion [1–22] is a most effective method in quantum mechanics and field theory, widely used at present in various physical calculations (e.g. see reviews [8–10]). Versions of this method differ in the choice of the expansion parameter: $N = l + D/2$, where $l = 0, 1, \ldots$ is the angular momentum and D the space dimension [3–5]; $N = l + D/2 - a$ (the shifted 1/N-expansion [7], with a the shift parameter); $N = [l(l+1)]^{1/2}$ (see [17]); etc. Here we discuss the version of this method introduced in [11] and developed in [12–22]. An essential aspect of this approach is the possibility of not only using in the case of a discrete spectrum but using it in calculating the energies and widths in Breit–Wigner resonances (complex energy $E = E_r - \mathrm{i}\Gamma/2$). This problem frequently occurs in atomic and nuclear physics, scattering theory, etc. As an example, we consider quasi-stationary states in short-range potentials, the Stark effect in a strong field, and a hydrogen atom in electric and magnetic fields. The structure of the higher-order terms in the 1/N-expansion and the divergence of this expansion are also discussed.

2. Description of the method

Consider the attractive potential

$$V(r) = -\frac{\hbar^2}{MR^2} g v(x) \qquad x = \frac{r}{R} \tag{2.1}$$

where g is the dimensionless coupling constant, R the characteristic range of the potential $V(r)$, $n = p + l + 1$ the principal quantum number, $p = 0, 1, 2, \ldots$ the radial quantum number (sometimes denoted by n_r), and l the angular momentum; below we set $\hbar = M = 1$. If $n \to \infty$ and p is fixed, the particle is localized around the classical equilibrium point $x = x_0$ determined by the following equation [11, 12]:

$$x^3 v'(x) = -\nu \tag{2.2}$$

with $\nu = n^2 g^{-1}$, and the energy E_{nl} is $(g^2/2n^2 R^2)\epsilon_{nl}$, with

$$\epsilon = \epsilon^{(0)} + \frac{\epsilon^{(1)}}{n} + \frac{\epsilon^{(2)}}{n^2} + \cdots. \tag{2.3}$$

Note that the coefficients of the $1/n$-expansion, $\epsilon^{(k)}$, depend on ν, or n (e.g. see equation (4.2) below). As a rule, the series (2.3) is not convergent but only asymptotic; nevertheless, in some simple cases it gets truncated. For example, $\epsilon^{(0)} = -1$ and $\epsilon^{(k)} = 0$ for $k \geq 1$ with the Coulomb potential.

In the case considered here, $D = 3$, $l \to \infty$, and $N \equiv n$. We will, therefore, speak of the $1/n$-expansion. The comparison of the Schrödinger equation at $r \sim r_0 = Rx_0$ with the equation describing the harmonic oscillator shows that n is analogous to $M/\hbar$. The amplitude of vacuum fluctuations of this oscillator is proportional to $(\hbar/M\omega)^{1/2} \sim n^{-1/2}$. Accordingly, we introduce

$$x = x_0(1 + \xi n^{-1/2}) \tag{2.4}$$

and expand all quantities in powers of $n^{-1/2}$ (contrary to r, the variable ξ remains finite as $n \to \infty$). In this way the coefficients of expansion (2.3) and the corresponding coefficients of the wavefunction expansion (in powers of $n^{-1/2}$) can be determined successively. For instance,

$$\epsilon^{(0)} = \frac{(1 - v_{-2})\nu^2}{x_0^2} \qquad \epsilon^{(1)} = \frac{(2p+1)(\omega - 1)\nu^2}{x_0^2}$$

$$\epsilon^{(2)} = \left(\frac{\nu}{\omega^2 x_0}\right)^2 \left\{ \frac{3(2p+1)^2}{2} \omega(\sigma - \omega^2) + \frac{1}{2}\omega^2(\tau + 1) \right. \tag{2.5}$$

$$\left. - \frac{11}{16}\sigma^2 + p(p+1)\left[\omega^4 + \omega^2(\tau - 1) - \frac{15}{8}\sigma^2\right] \right\}$$

where

$$\omega = [3(1 - v_0)]^{1/2} \qquad \sigma = 4(1 - v_1) \qquad \tau = \frac{9}{2} - \frac{15v_2}{2} \tag{2.6}$$

$$v_k = \frac{2(-x)^{k+1}}{(k+3)!} \frac{d^{k+2}v}{dx^{k+2}} \Big/ \frac{dv}{dx}\Big|_{x=x_0(\nu)} \tag{2.7}$$

and $v_{-2} = -2v(x_0)/x_0 v'(x_0)$ (note that ω is the frequency of small oscillations about the equilibrium point x_0). Analytical expressions for coefficients $\epsilon^{(k)}$ with $k \geq 3$ appear too cumbersome, but they can easily be obtained via recurrence relations convenient for a computer.

Special consideration is required in the case of $E = 0$ (i.e. in the event of a bound state appearing in a short-range potential). As a rule, the 'reduced' energy $\epsilon(\nu)$ increases with ν and at some value $\nu = \nu_{nl}^{cr}$ the nl-level moves into the continuous spectrum. In the lowest approximation ($n \rightarrow \infty$),

$$\nu_{cr} = 2\bar{x}_0^2 v(\bar{x}_0) \qquad \left.\frac{d\epsilon^{(0)}}{d\nu}\right|_{\nu=\nu_{cr}} = 2v(\bar{x}_0) \tag{2.8}$$

where $\bar{x}_0$ is the root of the equation $xv' + 2v = 0$. Corrections proportional to $1/n$ can be obtained from (2.5). Denoting the coupling constant corresponding to the zero energy of the nl-level by g_{nl}, we obtain

$$g_{nl} = \frac{n^2}{2\bar{x}_0^2 v(\bar{x}_0)} \left[1 + \frac{b_1}{n} + \frac{b_2}{n^2} + O\left(\frac{1}{n^3}\right) \right] \tag{2.9}$$

where

$$
\begin{aligned}
b_1 &= (2p+1)(\omega - 1) \\
b_2 &= \frac{1}{2}\left[\omega^2 - \omega - 5 + \frac{1}{\omega^2}(3\sigma + \tau + 3) - \frac{11\sigma^2}{8\omega^4} \right] \\
&\quad + p(p+1)\left[2\omega^2 - 2\omega - 9 + \frac{1}{\omega^2}(6\sigma + \tau + 3) - \frac{15\sigma^2}{8\omega^4} \right]
\end{aligned}
\tag{2.10}
$$

and the quantities ω, σ and τ are taken at $x = \bar{x}_0$.

With a further increase in ν the frequency ω vanishes at $\nu = \nu_*$. At this point the classical solution loses its stability—a collapse of two solutions occurs, one corresponding to the stable equilibrium point and the other to the unstable equilibrium point in the effective potential $U(r) = V(r) + l(l+1)/2r^2$, which includes the centrifugal energy. For $\nu > \nu_*$ the potential $U(r)$ possesses no minimum at r real, the equilibrium point moves into the complex plane and the coefficients of the $1/n$-expansion become complex-valued. But such solutions, though having no physical meaning in terms of classical mechanics, are of special importance to quantum mechanics: they are the solutions determining within the $1/n$-expansion not only the position but also the width of a resonance level.

3. Wavefunctions and the $\psi^2(0)$ value

Evaluating wavefunctions and, in particular, determining the coefficients of their asymptotics as $r \rightarrow 0$ and $r \rightarrow \infty$, is a problem of considerable importance to physics. Using only two terms of the $1/n$-expansion, we can obtain analytical expressions that are asymptotically exact as $n \rightarrow \infty$ for an arbitrary smooth potential.

Performing to this end the substitution (2.4), we take into account that in the region $|\xi| \ll n^{1/2}$ the function $\chi_{nl} = rR_{nl}(r)$ coincides with the wavefunction of the p-level of a harmonic oscillator with a frequency ω. Continuing this function (by the WKB method) into the sub-barrier region $\xi < \xi_-$ or $\xi > \xi_+$, where $\xi_\pm = \pm[(2p+1)/\omega]^{1/2}$, and expanding all quantities at $x = x_0$ in powers of $1/n$,

we come to the final result. Here we present only the expression for the coefficient of the asymptotics at the origin:

$$\chi_{nl}(r) = c_{nl}R^{-(l+3/2)}r^{l+1} + \ldots \qquad r \to 0 \tag{3.1}$$

$$\tilde{c}_{nl} = \left(\frac{n\omega^3}{\pi(p!)^2}\right)^{1/4}(2\pi\omega)^{p/2}x_0^{p-1/2}\exp\{-[nJ_0 + (2p+1)J_1]\} \tag{3.2}$$

(the tilde indicates that this expression is approximate and consists only of two terms of the $1/n$-expansion). Here

$$J_0 = \ln x_0 + \int_0^{x_0}dx\left[Q_0(x) - \frac{1}{x}\right] \qquad J_1 = \frac{1}{2}\int_0^{x_0}dx\left[\frac{x_0}{x(x_0 - x)} - \frac{S_0(x)}{Q_0(x)}\right]$$

$$\tag{3.3}$$

$$Q_0(x) = \left[\frac{1}{x^2} - \frac{2}{\nu}v(x) - \frac{\epsilon^{(0)}}{\nu^2}\right]^{1/2} \qquad S_0(x) = x^{-2} + (\omega - 1)x_0^{-2}$$

(for further details and examples we refer the reader to appendix 1).

Similar expressions can be obtained for $r > r_+$, including the asymptotic coefficient as $r \to \infty$ and the effective range r, [18, 19]. A simple analytical formula for the width of a quasi-stationary state can also be obtained (see equation (A2.5) in appendix 2).

Let us consider the wavefunction at finite r values, confining ourselves to the nodeless states for simplicity ($p = 0$ and $n = l + 1$). We have

$$\chi_{n,n-1} = \left(\frac{n\omega}{\pi x_0^2}\right)^{1/4}\exp\left(-\frac{1}{2}\omega\xi^2\right)\left[1 + \left(\frac{1}{3}a\xi^3 + b\xi\right)n^{-1/2} + O(n^{-1})\right] \tag{3.4}$$

where $a = 2(1 - v_1)\omega^{-1}$, $b = (a - 1)\omega^{-1}$ and $\int_0^\infty \chi^2\,dx = 1 + O(1/n)$. This expression is valid in the vicinity of the equilibrium point x_0, including the turning points $x_\pm = x_0(1 + n^{-1/2}\xi_\pm)$. The terms proportional to $n^{-1/2}$ and n^{-1} take into account the anharmonicity corrections and considerably improve the agreement of equation (3.4) with numerical calculations (see appendix 1).

Since the potential $V(r)$ enters into equations (3.2) and (3.3) explicitly, we can easily analyse how the $\psi(0)$ values depend on the potential modifications (e.g. see [18]).

4. A few model potentials

The above expressions are asymptotically exact as $n \to \infty$ and p is fixed. However, in applications the case if $n \sim 1$ is the most frequent. It is not obvious beforehand how useful the $1/n$-expansion is for small quantum numbers. Below we consider some examples.

(a) For power-like potentials

$$V(r) = \frac{gr^N}{N} \qquad N > -2 \tag{4.1}$$

we have $E_{nl}(g) \propto g^{2/(N+2)}$ and $c_{nl}(g) \propto g^{(l+3/2)/(N+2)}$; therefore, it is sufficient to put $g = 1$. This case corresponds to

$$v(x) = -\frac{x^N}{N} \qquad v_k = \frac{2(-1)^{k+1}(N-1)!}{(k+3)!(N-k-2)!} \qquad x_0 = n^{2/(N+2)}$$

and the energy expansion is

$$E_{nl} = \frac{1}{2} n^{2N/(N+2)} \left(\alpha_0 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots \right) \tag{4.2}$$

where $\alpha_0 = 1 + 2N^{-1}$, $\alpha_1 = (2p+1)/(\omega - 1)$ and

$$\alpha_2 = -\frac{(N-2)(\omega-1)}{48(N+2)} \left[(2p+1)^2(\omega^3 + \omega^2 - 12\omega + 12) - \frac{1}{3}(\omega^3 + \omega^2) \right] \tag{4.3}$$

with $\omega = (N+2)^{1/2}$, $\sigma = -\frac{1}{3}(N-5)(N+2)$ and $\tau = \frac{9}{2} + \frac{1}{8}(N-1)(N-2)(N-3)$ and $\epsilon^{(k)} = \alpha_k n^{4(N+1)/(N+2)}$ (note that the coefficients α_k are independent of the quantum number n). For $-1 \le N \le 4$ the coefficients α_k decrease rapidly as k increases. At $N = -1$ (the Coulomb potential) and $N = 2$ (the harmonic oscillator) all $\alpha_k \equiv 0$ for $k \ge 2$, and the series (2.3) is truncated and coincides with exact solutions:

$$\epsilon_{nl} = \begin{cases} -1 & \text{if } N = -1 \\ 2n^2(2p+l+3/2) & \text{if } N = 2. \end{cases} \tag{4.4}$$

Let us now investigate the accuracy of the asymptotics (3.2). For power-like potentials a comparison of equation (3.2) with numerical calculations is given in table 1. Even for the ground state $l = 0$ the accuracy of this simple expression is surprisingly high, especially at $N = 1$ and $N = 4$ (anharmonic oscillator). Note that in this case coefficients $c_{n,n-1}$ change by many orders of magnitude:

$$c_{n,n-1} \sim \frac{\text{const} \times a^n n^\beta}{(n\alpha)!} \qquad n \to \infty \tag{4.5}$$

where $\alpha = 2/(N+2)$, $\beta = (3N-2)/4(N+2)$ and $0.368 < a < 0.737$ for $-1 < N < \infty$. The factorial decrease in $c_{n,n-1}$ as n increases is due to the centrifugal barrier.

Table 1. The accuracy of the 1/n-expansion for $\psi(0)$ with power-like potentials.

l	$N = -1$	$N = 0$	$N = 1$	$N = 2$	$N = 4$	$N = 8$
0	1.0209	1.0280	0.9952	0.9803	0.99002	1.0882
1	1.0104	1.0146	0.9977	0.9898	0.99844	1.0731
2	1.0070	1.0098	0.9984	0.9931	1.00019	1.0598
3	1.0052	1.0074	0.9988	0.9948	1.00070	1.0502
5	1.0035	1.0050	0.9993	0.9965	1.00089	1.0378
10	1.0019	1.0027	0.9997	0.9981	1.00072	1.0232
20	1.0010	1.0014	0.9998	0.9990	1.00046	1.0131

Note: The values of the ratio $\tilde{c}_{nl}/c_{nl}$ ($n = l+1, p = 0$ states) are given in table 1. The coefficients $\tilde{c}_{nl}$ were calculated using equation (3.2) and the c_{nl} by solving the Schrödinger equation numerically. The case of $N = 0$ corresponds to the logarithmic potential $V(r) = g \ln r$.

Thus, for power-like potentials the accuracy of the asymptotics (3.2) with $p = 0$ is high in a wide range of N values but lowers as the number of nodes, p, increases [19].

(b) The funnel potential

$$V(r) = -\frac{\kappa}{r} + \frac{r}{a^2} \tag{4.6}$$

is frequently used to describe quarkonium and multiquark systems (e.g. see [23, 24]). Here $v(x) = x^{-1} - x$, $R = (\kappa a^2)^{1/2}$ and $\omega = \left[(1 + 3x_0^2)/(1 + x_0^2)\right]^{1/2}$, and x_0 is determined by the equation

$$x^3 + x = \nu \equiv \frac{n^2}{a\kappa^{3/2}}. \tag{4.7}$$

The first three terms in (2.3) are determined in this case explicitly [12]. A comparison of the $1/n$-expansion with the results of numerical computations is presented in table 2, where

$$\zeta_{nl}^{(k)} = \zeta^{(0)} + \zeta^{(1)}\frac{1}{n} + \ldots + \zeta^{(k)}\frac{1}{n^k}. \tag{4.8}$$

The Schrödinger equation with the potential (4.6) was preliminarily transformed in advance ($r = \text{const} \times \rho$) to the standard form

$$\frac{d^2 u_{nl}}{d\rho^2} + \left[\zeta + \frac{\lambda}{\rho} - \rho - \frac{l(l+1)}{\rho^2}\right] u_{nl} = 0 \tag{4.9}$$

which determines the eigenvalues $\zeta = \zeta_{nl}(\lambda) = 2^{1/3} a^{4/3} E_{nl}$. Here

$$\int_0^\infty u_{nl}^2(\rho)\,d\rho = 1 \qquad u_{nl}(\rho) = c_{nl}\rho^{l+1} + \ldots \quad \text{as} \quad \rho \to 0. \tag{4.10}$$

Table 2. The accuracy of the $1/n$-expansion for the lower levels in the funnel potential.

k	1S $(n = 1)$	1P $(n = 2)$	2S $(n = 2)$	1D $(n = 3)$	2P $(n = 3)$	3S $(n = 3)$
0	0.605 23	2.269 24	2.269 24	3.387 63	3.387 63	3.387 63
1	0.996 34	2.617 17	3.313 05	3.368 76	4.321 03	4.943 30
2	0.979 32	2.611 44	3.264 76	3.695 78	4.302 75	4.894 42
3	0.979 85	2.611 01	3.246 27	3.695 57	4.298 71	4.876 27
5	0.980 29	2.611 14	3.231 82	3.695 60	4.296 92	4.856 51
exact ζ_{nl}	0.980 37	2.611 13	3.228 85	3.695 60	4.296 70	4.842 09
c_{nl}	2.0833	0.8970	1.6634	0.3103	1.0701	1.5362
$\tilde{c}_{nl}/c_{nl}$	0.9977	0.9976	1.40	0.9983	1.22	

Note: The values of $\zeta_{nl}^{(k)}$ (see equation (4.8)) are given for $2M = m_c = 1.84\,\text{GeV}$, $\kappa = 0.52$ and $a = 2.34\,\text{GeV}^{-1}$, which corresponds to the Coulomb parameter $\lambda = \kappa(2Ma)^{2/3} = 1.376\,23$. We obtained the exact eigenvalues ζ by solving the Schrödinger equation numerically (see also [25]). The coefficients c_{nl} for $\rho \to 0$ correspond to the normalization condition in (4.10).

We see that the description of the energy spectrum using the partial sums of the $1/n$-expansion is fairly accurate for the funnel potential. The accuracy grows with l and it is especially high for nodeless states ($1S, 2P, \ldots$). In the latter case only three terms of the $1/n$-expansion ensure an accuracy of one per cent in the energy and

$\psi(0)$ computations and properly reproduce the charmonium spectrum if equation (4.6) with Cornell potential parameters [23] is used. What is important is that the above expressions make it possible to easily perform the calculations for other potentials arising in QCD.

(c) For the Yukawa potential, $V(r) = -r^{-1}\exp(-\mu r)$, we have $v(x) = e^{-x}x^{-1}$ and $\nu = n^2\mu$, with μ the screening parameter. The dependence of x_0 and $\epsilon^{(0)}$ on ν is determined from the equations:

$$\nu = (x^2 + x)e^{-x} \qquad \epsilon^{(0)} = (x^2 - 1)e^{-2x}. \tag{4.11}$$

In this case $\nu_{\mathrm{cr}} = 2e^{-1} = 0.736$ and $\nu_* = 0.834$. As $\nu \to \nu_*$ the coefficients $\epsilon^{(k)}$ acquire singularities,

$$
\begin{aligned}
\epsilon^{(0)} &= c_0 + c_1(\nu - \nu_*) + c_2(\nu_* - \nu)^{3/2} + \ldots \\
\epsilon^{(1)} &= c_0' + c_1'(\nu_* - \nu)^{1/4} + \ldots \qquad \epsilon^{(2)} \propto (\nu_* - \nu)^{-1}
\end{aligned}
\tag{4.12}
$$

with $\epsilon^{(k)}(\nu \to \nu_*) \to \infty$ for $k \geq 2$. Therefore, the $1/n$-expansion is of no use in the neighbourhood of $\nu = \nu_*$. However, for $\nu > \nu_*$, when the equilibrium point $x_0(\nu)$ moves into the complex plane, the $1/n$-expansion is applicable again. As in the case of the funnel potential, only three terms of the series (2.3) provide an acceptable accuracy in energy calculations for a quasi-stationary state, and the greater the value of l the higher the accuracy (see figures 1–3).
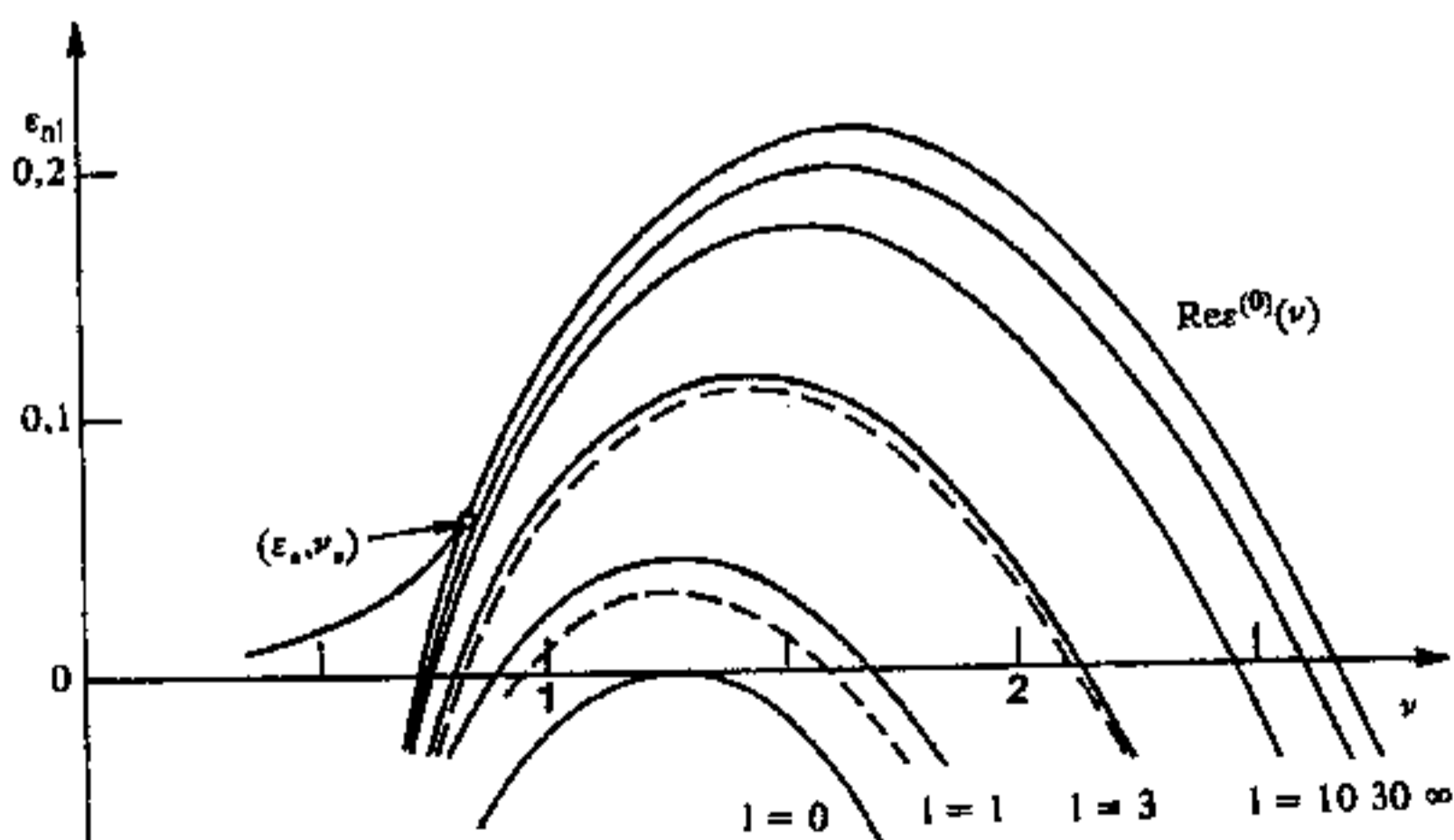

Figure 1. The Yukawa potential ($\nu_* = 0.83996$ and $\epsilon_* = 0.0636$). The reduced energies ϵ_{nl} refer to nodeless states with $l = n - 1$. The full curves correspond to exact solution and the broken curves to the first two terms in the $1/n$-expansion. At $l = 10$ and $l = 30$ these curves coincide with the full curves (on the scale of this figure).
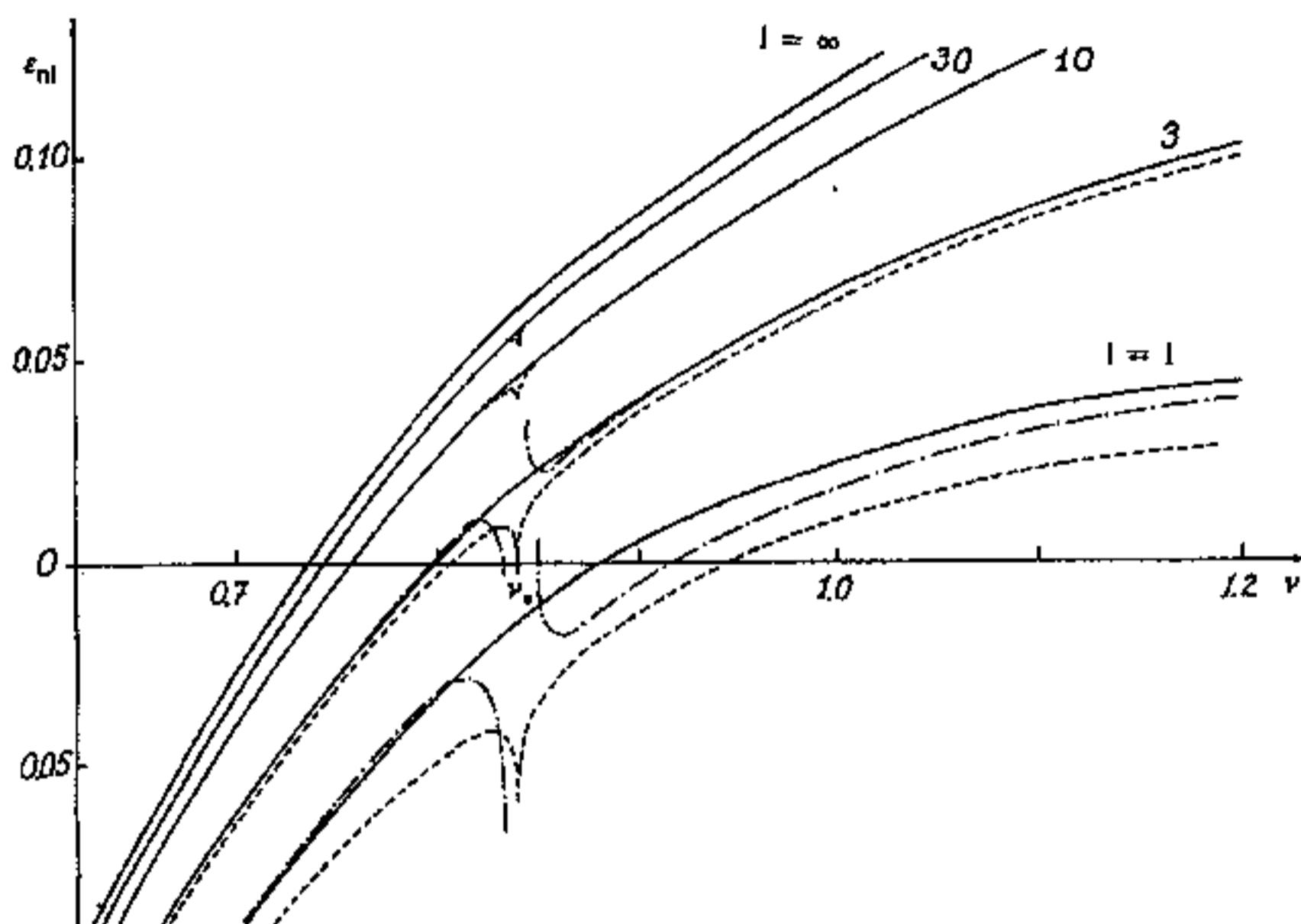
Figure 2. The same as in figure 1 near $\nu = \nu_*$. The full, broken and dotted curves correspond to the sum of one, two and three terms of the $1/n$-expansion (2.3).

(d) The exponential potential is frequently used in nuclear physics. In this case $v(x) = e^{-x}$ and the equations are

$$\nu = x^3 e^{-x} \qquad \epsilon^{(0)} = (x^4 - 2x^3) e^{-2x} \tag{4.13}$$

and $\nu_{cr} = 1.083$ and $\nu_* = 1.344$. The energy curves are similar to those in figure 1.

(e) For an arbitrary screened Coulomb potential

$$V(r) = -\frac{f(x)}{r} \qquad f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f_k x^k \tag{4.14}$$

with $x = \mu r$, $f(x) \equiv x v(x)$, $f_0 = 1$ and μ the screening parameter†, as $\nu \to 0$, we get

$$\begin{aligned}
x_0 &= \nu + \frac{1}{2} f_2 \nu^3 - \frac{1}{3} f_3 \nu^4 + \frac{1}{8}(f_4 + 6f_2^2)\nu^5 + \ldots \\
\omega &= 1 - \frac{1}{2} f_2 \nu^2 + \frac{1}{2} f_3 \nu^3 - \frac{1}{8}(2f_4 + 7f_2^2)\nu^4 + \ldots \\
\epsilon^{(0)} &= -1 + 2f_1 \nu - f_2 \nu^2 + \frac{1}{3} f_3 \nu^3
\end{aligned} \tag{4.15}$$

† Actually, we can write any potential $V(r)$ in this form (cf equations (4.14) and (2.1), where $R = \mu^{-1}$ and $g = M/\mu$). For one thing, the screening functions $f(x) = e^{-x}$, $x/(e^x - 1)$, $(1+x)^{-2}$ and $(1-x^2)$ correspond to the Yukawa, Hülthen, Tietz and funnel potentials. However, we assume in subsequent formulae that the potential is Coulomb-like as $r \to 0$, since we set $f(0) = f_0 = 1$.
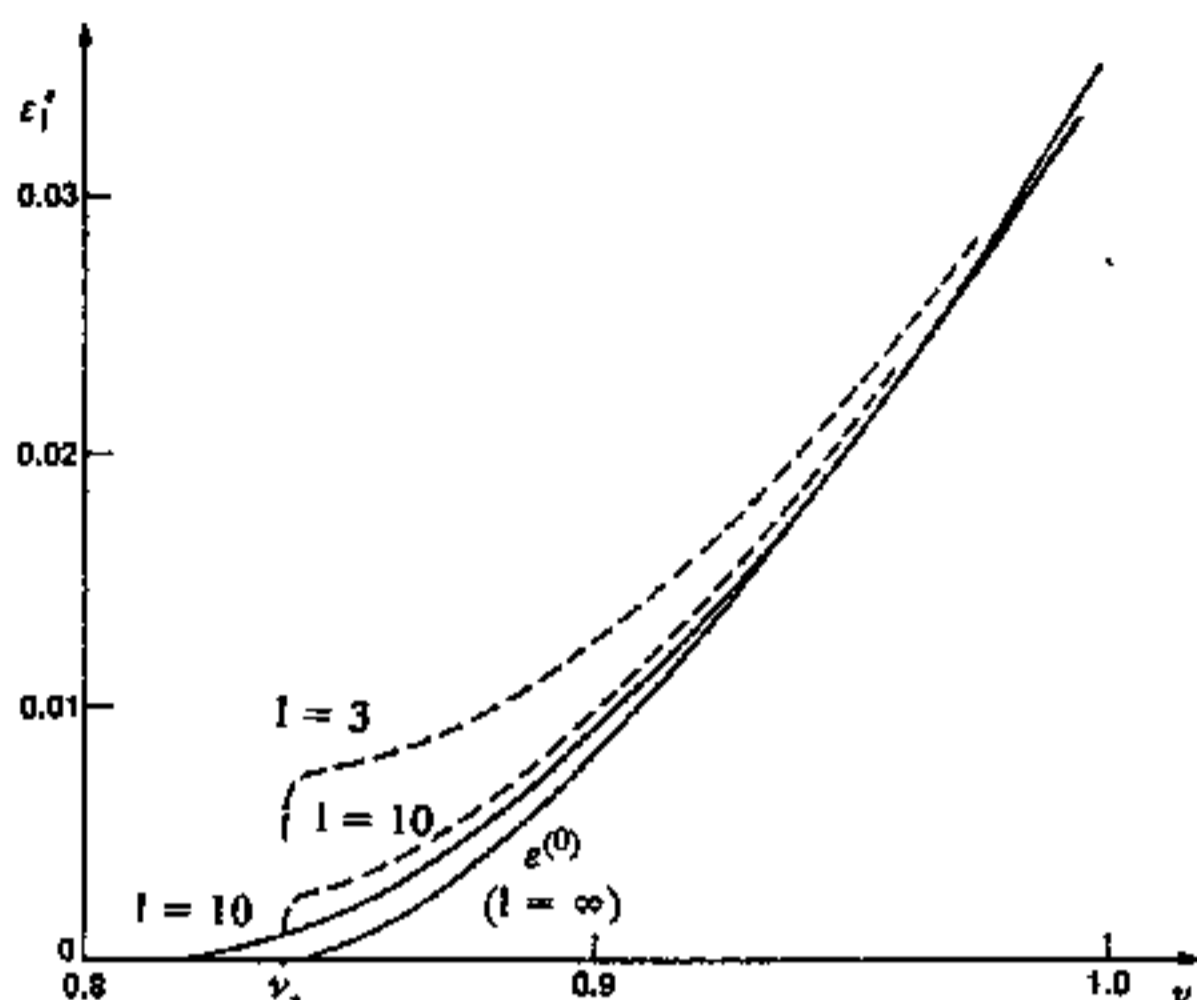
Figure 3. Values of $\epsilon_l'' = n^2 \Gamma_{nl}$ for states with $l = n - 1$ in the Yukawa potential. The full curve for $l = 10$ corresponds to the exact solution, the broken curves ($l = 3$ and 10) to the first two terms of the expansion and the full curve for $\epsilon^{(0)}$ to $l = \infty$.

$$- \frac{1}{12} \left(f_4 + 3 f_2^2 \right) \nu^4 + \frac{1}{60} \left(f_5 + 20 f_2 f_3 \right) \nu^5 + \dots \tag{4.16}$$

$$\epsilon^{(1)} = -\frac{1}{2} f_2 \nu^2 + \frac{1}{2} f_3 \nu^3 - \frac{1}{8} \left(2 f_4 + 3 f_2^2 \right) \nu^4 + \dots$$

$$\epsilon^{(2)} = \frac{1}{6} f_3 \nu^3 - \frac{1}{48} \left(11 f_4 + 12 f_2^2 \right) \nu^4 + \dots \qquad \epsilon^{(3)} = -\frac{1}{16} f_4 \nu^4 + \dots .$$

Since $\epsilon^{(k)}(\nu)$ is proportional to $f_{k+1} \, \nu^{k+1}$ for $k \geq 1$ and as $\nu \to 0$, the $1/n$-expansion rapidly converges in this region. On the other hand, at $\nu \simeq \nu_*$,

$$\omega = c \, \delta^{1/4} + \dots \qquad x_{0,1} = x_* [1 \mp b \, \delta^{1/2} + O(\delta)] \tag{4.17}$$

where

$$c = \left[\frac{2(x^3 f''' + 3 x^2 f'')}{f - x f'} \right]^{1/4} \equiv \left(-\frac{2 x^2 \nu''}{\nu} \right)^{1/4} \tag{4.18}$$

$\delta = 1 - \nu/\nu_* \to 0$, $\nu = x f - x^2 f'$ and all functions (f, $f' \equiv \mathrm{d}f/\mathrm{d}x$, etc) are taken at point $x = x_*$ determined by the equation

$$f - x f' - x^2 f'' = 0 \qquad \text{or} \quad \nu' = 0. \tag{4.19}$$

The coefficients of the singular terms in equations (4.12) and (4.17) are closely related to each other: $b = 2c^{-2}$ and $c_2 = 2c^6(\nu_*/x_*^6)^{3/2}$. We also note that

$$\frac{\mathrm{d}\epsilon^{(0)}}{\mathrm{d}\nu} = -2 f'(x_0) \qquad \frac{\mathrm{d}^2 \epsilon^{(0)}}{\mathrm{d}\nu^2} = - \frac{2 f''}{f - x f' - x^2 f''} \bigg|_{x=x_0} . \tag{4.20}$$

From equations (4.12) and (4.20) it follows that $\nu = \nu_*$ is a singular point for the function $\epsilon^{(0)}(\nu)$ since $d^2\epsilon^{(0)}/d\nu^2 \to \infty$. The quantities x_0 and x_1 in (4.17) correspond to the stable and unstable equilibrium points in the effective potential, which includes the centrifugal energy. These points coincide at $\nu = \nu_*$ and then, for $\nu > \nu_*$, move into the complex plane. Such a solution has no physical meaning in classical mechanics, but in quantum mechanics it allows calculating, within the framework of the $1/n$-expansion, the position and width of Breit–Wigner resonances.

5. The Stark effect in a strong field

We discuss here the application of the $1/n$-expansion to this problem, restricting ourselves to $|0, 0, n-1\rangle$ states of a hydrogen atom. The states have no 'radial' excitations (in parabolic coordinates ξ and η). In the limit of $n \to \infty$ the classical equilibrium point (ξ_0, η_0) and the energy E are determined by the system of equations

$$\frac{dU_1}{d\xi} = \frac{dU_2}{d\eta} = 0 \qquad U_1(\xi_0) = U_2(\eta_0) = \frac{1}{4}E \qquad \beta_1 + \beta_2 = 1 \qquad (5.1)$$

where $\xi = r + z$ and $\eta = r - z$ are the parabolic coordinates, β_i the separation constants, and U_i the effective potentials [26]:

$$U_1(\xi) = -\frac{\beta_1}{2\xi} + \frac{m^2-1}{8\xi^2} + \frac{1}{8}\mathcal{E}\xi \qquad U_2(\eta) = -\frac{\beta_2}{2\eta} + \frac{m^2-1}{8\eta^2} - \frac{1}{8}\mathcal{E}\eta \qquad (5.2)$$

(we use atomic units). Substituting equations (5.2) into (5.1), we arrive at a system of five equations. Their solution can be considerably simplified if we take into account that for $n \gg 1$ the Bohr atom model is applicable. The state $|0, 0, m\rangle$ corresponds to a circular electron orbit perpendicular to the z axis. When an electric field $\mathcal{E}$ is turned on, the orbit is shifted along the z axis and changes its radius, remaining circular. From the conditions of equilibrium of forces acting on the electron in its rest frame we get

$$\mathcal{E} = -zr^{-3} \qquad \rho r^{-3} = v^2\rho^{-1} \qquad \rho v = m \approx n \qquad E = \tfrac{1}{2}v^2 - r^{-1} + \mathcal{E}z \qquad (5.3)$$

where $\rho = (r^2 - z^2)^{1/2}$. Performing the scaling transformation

$$r \to n^2 r \qquad v \to \frac{v}{n} \qquad \epsilon = \epsilon' - i\epsilon'' = 2n^2 E \qquad F = n^4\mathcal{E} \qquad (5.4)$$

and substituting $r = n^2(1 - \tau^2)^{-2}$, we finally arrive at the equations:

$$\tau(1 - \tau^2)^4 = F \qquad (5.5)$$

$$\epsilon^{(0)} = -(1 - \tau^2)^2(1 + 3\tau^2) \qquad (5.6)$$

$$\epsilon^{(1)} = (1 - \tau^2)^3[(1 + 3\tau)^{1/2} + (1 - 3\tau)^{1/2} - 2].$$

The higher-order coefficients $\epsilon^{(k)}$ can, in principle, be found analytically, but the expression for $\epsilon^{(2)}$ is already cumbersome [14]. We have calculated $\epsilon^{(k)}$ numerically via the recurrence relations described in [17]. As long as $0 < \tau < 1/3$ (or

$F < F_* = 2^{12}3^{-9} = 0.2081$), all the $\epsilon^{(k)}$ are real. In this region the $1/n$-expansion determines only the Stark shifts of the levels. For $F > F_*$ the root of equation (5.5) moves into the complex plane:

$$\tau = \frac{1}{3} \pm \mathrm{id}(F - F_*)^{1/2} + \ldots \qquad \epsilon^{(0)} = \epsilon_* + c_1 f - c_2 f^{3/2} + \ldots \quad (5.7)$$

where $f = 1 - F/F_* \to 0$, $d = 3^{5/2}2^{-9/2} = 0.6889$, $\epsilon_* = -2^8 3^{-5} = -1.0535$, $c_1 = 0.0176$ and $c_2 = 0.0165$. This threshold singularity is connected with the collapse of two equilibrium points (stable and unstable) and corresponds to barrier disappearance in the effective potential $U_2(\eta)$. Since the $\epsilon^{(k)}(F)$ are complex-valued for $F > F_*$, this offers the possibility of calculating within the $1/n$-expansion not only the shifts but also the widths of atomic levels.

Note that the widths of atomic states for $F < F_*$ can also be found if we continue the wavefunction of the type (3.4) into the sub-barrier region and calculate the barrier penetrability. Here we give the final result for the width of the $|0, 0, n - 1\rangle$ state:

$$\Gamma_n = \frac{2^{2n}n^{n-3}}{(n-1)!}\, p(\mathcal{E}) \exp\{-n\phi(\mathcal{E})\} \tag{5.8}$$

where $\Gamma_n \equiv \Gamma^{(0,0,n-1)}(\mathcal{E})$,

$$\phi = 2\left[z + \frac{z^3}{3(1 - z^2)} - \tanh^{-1} z \right] + 1 + 2\ln 2$$

the pre-exponential factor is

$$p(\mathcal{E}) = z^{7/2}\left(\frac{2}{1+z}\right)^2 (1 - \tau)(1 - \tau^2)^3$$

$$\times \exp\left\{ -\frac{(1 - 3\tau)^{1/2}}{\tau}[2 - (1 + 3\tau)^{1/2} - (1 - 3\tau)^{1/2}] \right\} \tag{5.9}$$

$$z = (1 - 3\tau)^{1/2}(1 - \tau)^{-1} \qquad 0 < z < 1$$

and $\tau = \tau(F)$ is determined from equation (5.5).

In the weak-field region, $F \ll F_*$, we have the following formula:

$$\Gamma_n = \frac{(4n)^n}{n!\, n^2} \exp\left\{ -n\left(\frac{2}{3F} + \ln F + \frac{11}{4} F \right) - \frac{9}{2} F \right\}\left[1 - \frac{5F}{3n} + O(F^2) \right] \tag{5.10}$$

which is asymptotically exact when $F \to 0$. Taking into account expansions (A2.14) and (A2.15) from appendix 2, we can easily see that, as $F \to 0$, equation (5.8) assumes the form of equation (5.10) with an accuracy of $O(1/n)$.

The calculated complex energy eigenvalues $E = E_r - i\Gamma/2$ for different states $|n_1, n_2, m\rangle$ of a hydrogen atom are listed in table 3, where $1/n$ denotes the results of the $1/n$-expansion and HPA marks the values obtained by perturbation series summation via Hermite–Padé approximations [15]. The agreement between different computational methods is fairly good.

The results for $|0, 0, n - 1\rangle$ Stark resonances are presented in figures 4 and 5, where the curve $n = \infty$ corresponds to the first term $\epsilon^{(0)}$ in the $1/n$-expansion.

Table 3. Energies and widths of the hydrogenic states $|n_1, n_2, m\rangle$ in an electric field $\mathcal{E}$.

		$\|0,0,n-1\rangle$		
n	F	$-\epsilon_n$		Computational method
1	0.1	$1.05484 + i0.01454$		HPA
		$1.054836 + i0.014538$		[28]
1	0.5	$1.25 + i0.55$		HPA
		$1.246 + i0.549$		$1/n$
3	0.5	$1.224 + i0.317$		HPA
		$1.22393 + i0.31685$		$1/n$
1	1.0	$1.248 + i1.294$		$1/n$
		$1.2487 + i1.2936$		[29]
3	1.0	$1.27891 + i0.83875$		$1/n$
10	1.0	$1.2851 + i0.6739$		HPA
		$1.28518 + i0.67388$		$1/n$

		$\left\|\frac{n-1}{2}, \frac{n-1}{2}, 0\right\rangle$ states			
n	$\mathcal{E}$ (a.u.)	F	ν	$10^6\Gamma$	Computational method
5	1.8×10^{-4}	0.1125	4.92402	2.283	HPA
			4.9239	2.22	$1/n$
			4.9240	2.282	[30]
11	1.0×10^{-5}	0.1464	10.713	2.83	HPA
			10.7128	2.82	$1/n$
			10.688	2.815	[30]
15	3×10^{-6}	0.1519	14.577	1.35	HPA
			14.5766	1.338	$1/n$
			14.5771	1.338	[30]

Note: Here $\epsilon = 2n^2(E_r - i\Gamma/2)$, $F = n^4\mathcal{E}$, $\hbar = e = m_e \approx 1$ and $\nu = (-2E_r)^{-1/2}$; for a free atom ($\mathcal{E} = 0$), $\nu = n$.

Note that in very strong fields the Stark shifts change sign. For further details we refer the reader to [14–16, 27], where the complex-valued energies $E = E_r - i\Gamma/2$ for different $|n_1, n_2, m\rangle$ states were calculated and compared with the experimental data on the photoionization spectra of H, Rb and Na atoms (for one thing, scaling relations for the Stark resonances with energies close to the ionization threshold $E = 0$ were obtained in [27]). The dependence of the Stark widths on the electric field strength is almost linear for $F > F_*$ (see figure 5 and [13–16]). A theoretical explanation of this is given in Appendix 3.

Other theoretical approaches to the problem of Stark resonances in the presence of a static electric field have been developed in [28–34]. The recent paper [34] contains a detailed analysis of the photoionization cross section of atomic hydrogen in terms of the sum over resonance contributions.
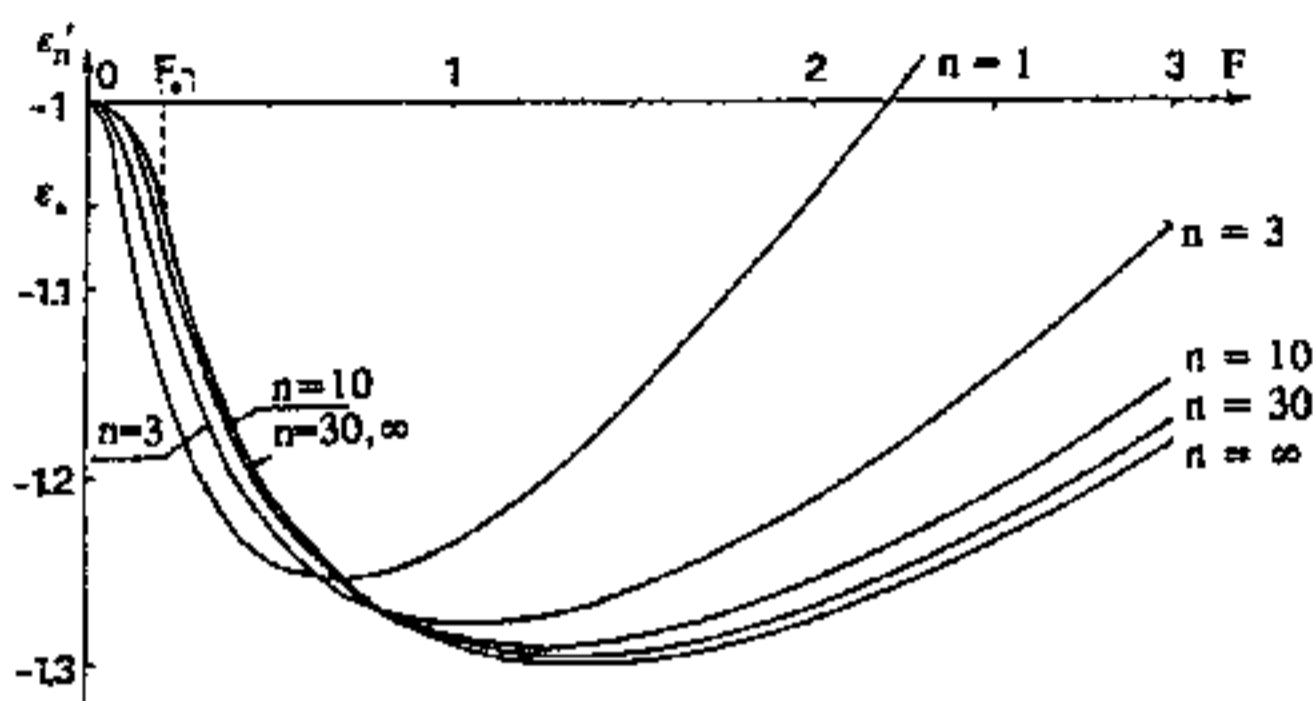
Figure 4. Stark shifts for the $|0,0,n-1\rangle$ states of a hydrogen atom, $\epsilon_n' = 2n^2\,\mathrm{Re}\,E^{(0,0,n-1)}$. We use the reduced variables (5.4), and $F_* = 0.2081$ is the classical ionization threshold [14].
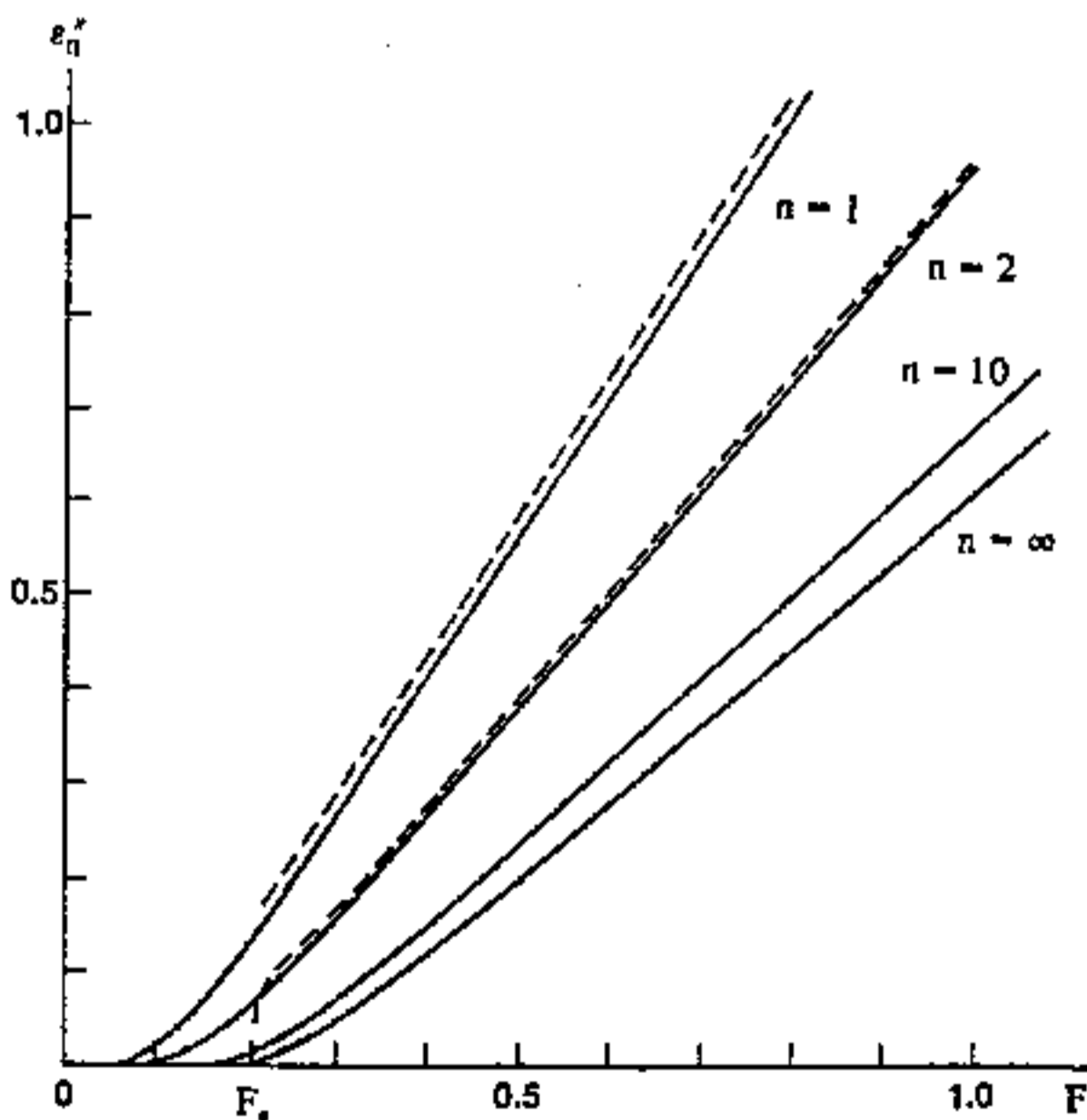

Figure 5. The electric field dependence of the reduced widths, $\Gamma^{(0,0,n-1)} = n^{-2}\epsilon_n''$, for the $|0,0,n-1\rangle$ states in hydrogen. Full curves correspond to exact solution [15], and broken curves to the sum of two terms, $\epsilon^{(0)}+\epsilon^{(1)}n^{-1}$, of the $1/n$-expansion (2.3). For $n \geq 10$ the broken curves coincide with the full ones.

6. A hydrogen atom in external fields

Here we restrict ourselves to the case of parallel electric $(\mathcal{E})$ and magnetic (H) fields

and to states with $m = n - 1$, that is, states with the maximum possible value of the magnetic quantum number m. The parameters of the classical electron orbit are determined from equilibrium conditions similar to (5.1) and from the quantization of angular momentum:

$$\frac{1}{2\pi} \oint (p_\varphi - A_\varphi)\rho \, \mathrm{d}\varphi = m\hbar \qquad p = v + \tfrac{1}{2} H \times r. \tag{6.1}$$

By means of the scaling transformation

$$F = n^4 \mathcal{E} \qquad B = n^3 H \qquad v = \tilde{v}/n \qquad r = n^2 \tilde{r} \qquad \epsilon = 2n^2 E$$

we introduce reduced quantities F, B, ϵ, etc that remain finite as $n \to \infty$ (the tilde over r and v is omitted below for simplicity). The equations for the classical electron orbit are

$$F = -zr^{-3} \qquad \rho r^{-3} - Bv = v^2 \rho^{-1} \qquad \rho(v + \tfrac{1}{2} B\rho) = 1$$

$$\rho = (r^2 - z^2)^{1/2} = 4(1 - F^2 r^4)^{1/2}. \tag{6.2}$$

Excluding velocity v, we obtain the equation that determines the radius of the equilibrium orbit, $r = r_0(F, B)$:

$$r(1 - F^2 r^4)^2 \left(1 + \tfrac{1}{4} B^2 r^3\right) = 1. \tag{6.3}$$

The first term in the $1/n$-expansion, $\epsilon^{(0)}$, is equal to the energy of the electron in this orbit, and the second is determined by small oscillations about it,

$$\epsilon^{(0)} = 2U(r_0) = \frac{2}{\rho_0^2} + \frac{\rho_0^2}{r_0^3} - \frac{4}{r_0} \tag{6.4}$$

$$\epsilon^{(1)} = (2n_1 + 1)\omega_1 + (2n_2 + 1)\omega_2 - 2(n_1 + n_2 + 1)\rho_0^{-2}$$

where $n_1 + n_2 + 1 = n - |m|$, $n_i = 0, 1, \ldots$ are oscillator quantum numbers, and the frequencies are

$$\omega_{1,2} = \left[\frac{1}{r_0^3} + \frac{1}{2} B^2 \pm \left(\frac{9F^2}{r_0^2} + 3F^2 B^2 r_0 + \frac{1}{4} B^4 \right)^{1/2} \right]^{1/2}. \tag{6.5}$$

Higher-order terms in the $1/n$-expansion ($k \geq 2$) take into account the anharmonicity of the effective potential $U(r)$ and are computed via recurrence relations. The effective potential $U(r)$ has a minimum only for fairly weak electric fields, $F < F_*(B)$. At $F = F_*$ the frequency ω_2 tends to 0, which corresponds to the collapse of two classical solutions. Therefore, in this case equation (6.3) has a double root, and the parametric equation for $F_* = F_*(B)$ is

$$F_* = F_*(0)(1 + 2t)^{1/2}\left(1 - \frac{t}{4}\right)^6 \frac{1}{(1-t)^2}$$

$$B = 2^9 3^{-11/2} t^{1/2} \left(1 - \frac{t}{4}\right)^{9/2} \frac{1}{(1-t)^2} \tag{6.5}$$
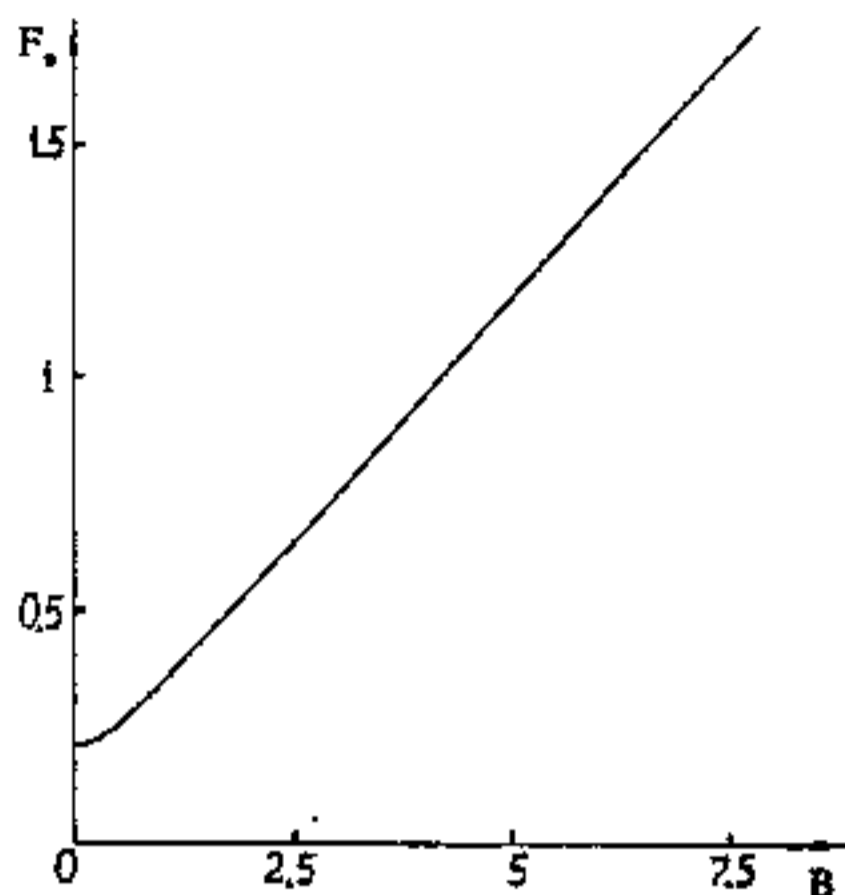
Figure 6. The classical ionization threshold $F_*(B)$ (atomic units).

with $0 < t < 1$. Here $t = 0$ corresponds to $B = 0$ and $F_*(0) = 2^{12}3^{-9}$ coincides with the value of the classical ionization threshold for the Stark problem (see the preceding section). Numerical calculations involving equations (6.6) give the curve in figure 6. For $F > F_*(B)$ the radius of the classical orbit and the coefficients $\epsilon^{(k)}$ become complex-valued.

Some of the results obtained are presented at figure 7 (for states with $n_1 = n_2 = 0$ and $m = n - 1$). Full curves represent the real and imaginary parts of the reduced energy. The real and imaginary parts of the classical energy $\epsilon^{(0)}(F)$ are shown by broken lines. We see that qualitatively $\mathrm{Re}\,\epsilon^{(0)}$ represents the dependence of the energy on the field. As for $\mathrm{Im}\,\epsilon^{(0)}$, it is a rough approximation, that is, summation of series (2.3) is necessary in this case.

The above examples clearly show that the $1/n$-expansion is an effective calculation method, especially for Rydberg states ($n \gg 1$). The investigation of such states, including those in external fields, is a problem of interest to atomic physics at present. Due to the development of laser techniques, the region of strong fields, which are comparable with the atomic field inside an electron orbit, becomes accessible. In these cases the $1/n$-expansion is the most suitable method, since it does not imply that the perturbation of the free atomic Hamiltonian is small.

7. The $1/n$-expansion and coherent states

As seen from the examples considered, the $1/n$-expansion has a sufficiently high accuracy for $n \sim 1$ (in the case of nodeless states, $p = 0$). Let us give a qualitative explanation of this fact. It will be shown that the above states are closest to classical mechanics. Since the first term $\epsilon^{(0)}$ in (2.3) corresponds to a classical particle at rest at the minimum of the effective potential, the high accuracy of the $1/n$-expansion is thus explained by the suitable choice of the initial approximation.

For states with $l = |m| = n - 1$ in a hydrogen atom we have [26] $\bar{r} = n(n + 1/2)$
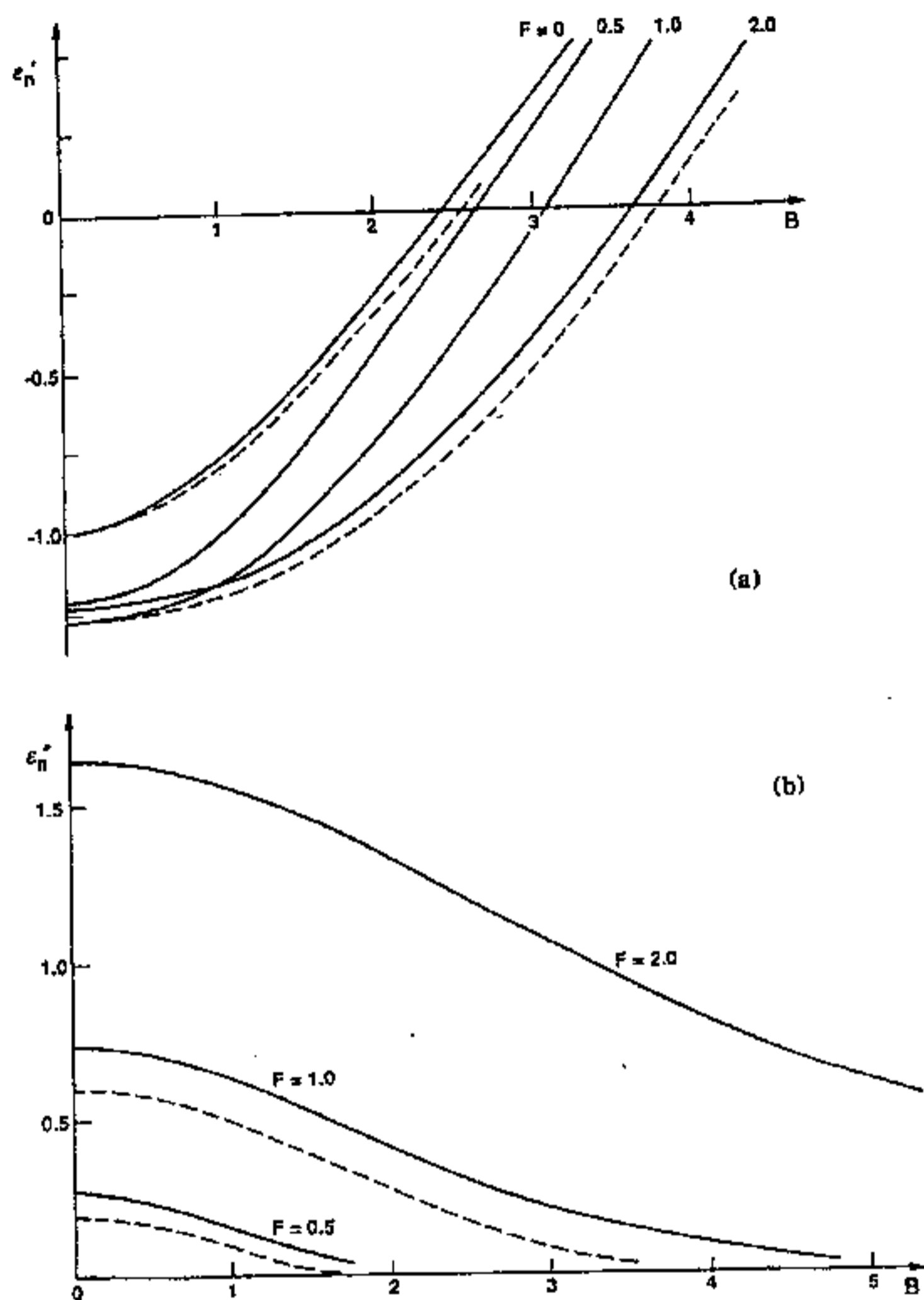
Figure 7. A hydrogen atom in parallel external fields: (a) the values of $\epsilon'_n = 2n^2 \operatorname{Re} E_n$, and (b) $\epsilon''_n = n^2 \Gamma_n$ for the state with $n_1 = n_2 = 0$ and $n = m + 1 = 10$. The values of the reduced electric field $F = n^4 \mathcal{E}$ are placed at the curves ($\hbar = e = m_e = 1$, atomic units).

and $\bar{p}_r^2 = [(2n - 1)n^2]^{-2}$. Hence,

$$\Delta p_r \, \Delta r = \frac{\hbar}{2}\left(\frac{2n+1}{2n-1}\right)^{1/2} \qquad \Delta p_z \, \Delta z = \frac{\hbar}{2}\left(\frac{2n+2}{2n+1}\right)^{1/2}. \qquad (7.1)$$

For $n \gg 1$ such a state, corresponding to a circular electron orbit perpendicular to the z axis, minimizes the uncertainty relations of the radial and transverse components (in relation to the orbit plane) of p and r.

For all other states, $\Delta p_i \, \Delta q_i$ exceeds $\hbar/2$. Thus, if $n \to \infty$ and the quantum

numbers $p \equiv n_r$ and $q = l - |m|$ are fixed,

$$\Delta p_r \, \Delta r = \hbar \left(p + \frac{1}{2} + \frac{a_1}{n} + \ldots \right) \qquad \Delta p_z \, \Delta z = \hbar \left(q + \frac{1}{2} + \frac{b_1}{n} + \ldots \right) \qquad (7.2)$$

where $a_1 = \frac{1}{4}[1 - 3p(p+1)]$ and $b_1 = \frac{5}{4}(2pq + p + q) - \frac{1}{8}(4q^2 - 1)$. If, on the other hand, p and q are both large, then $\Delta p_i \, \Delta q_i \propto n$. For example, for nl-states in the potential (4.1) we have $\Delta p_r \sim n^{N/(N+2)}$ and $\Delta r \sim n^{2/(N+2)}$, and

$$\Delta p_r \, \Delta r = \hbar n f(\rho) \qquad \rho = [l(l+1)]^{1/2} n^{-1} \qquad (7.3)$$

for $n \gg 1$ and $0 \le \rho < 1$. The results of numerical computations of $f(\rho)$ for the power potentials (4.1) are given in figure 8, which shows that as $n \to \infty$ the uncertainty product $\Delta p \, \Delta q \sim \hbar$ only for l close to $l_{\max} = n - 1$, that is, at $p \sim 1$. This makes the semiclassical approach to such states natural. In other cases the quantum fluctuations of the orbit radius increase without limit as $n \to \infty$, with the result that the notion of a classical orbit becomes meaningless.
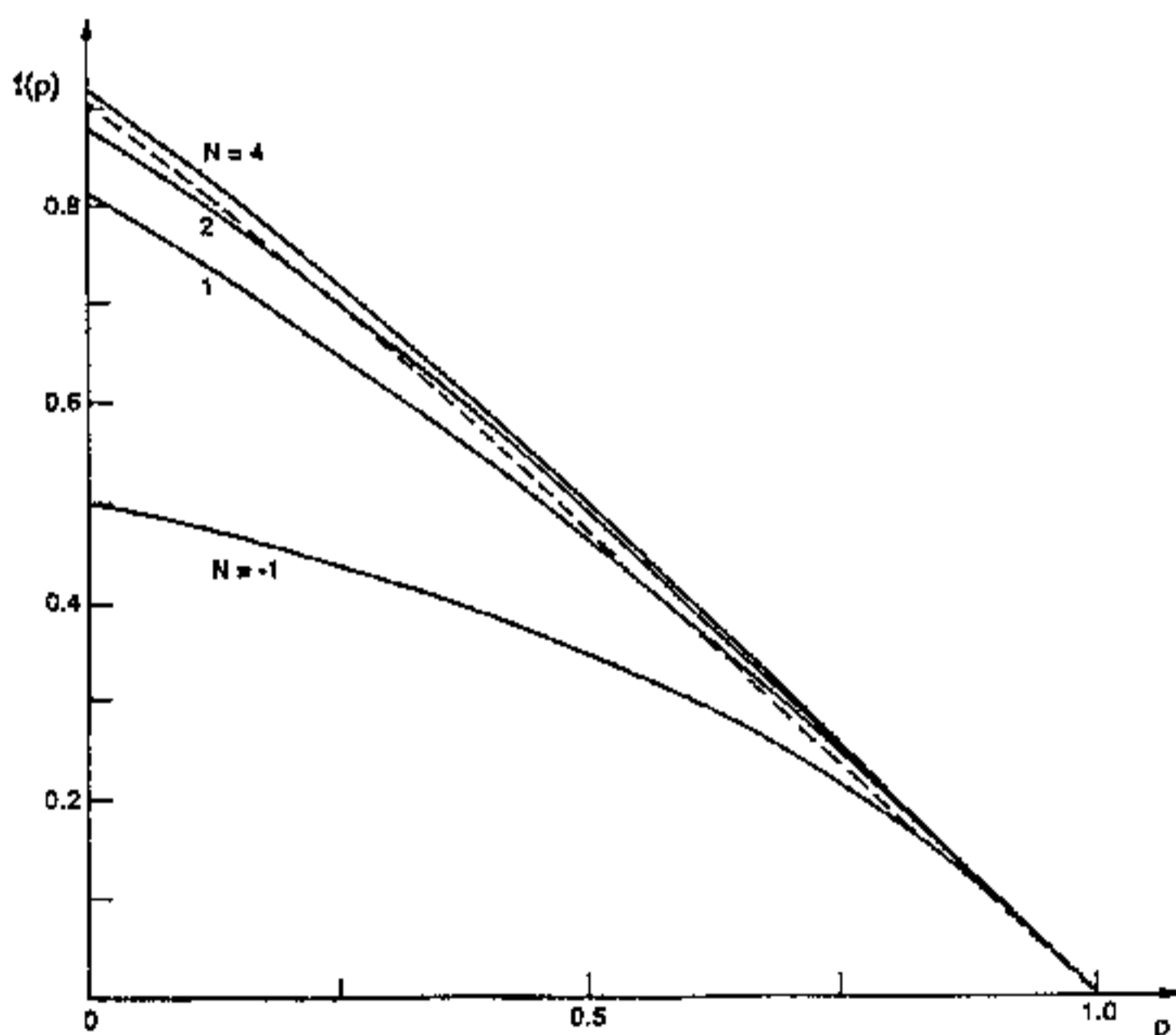

Figure 8. The function $f(\rho)$ in equation (7.3) for power potentials. The values of N are placed at the curves. The broken curve corresponds to $N = \infty$, that is, the square-well potential.

In conclusion the following should be noted. For the harmonic oscillator the Glauber coherent states $|\alpha\rangle$ were introduced [35] for which $\Delta p \, \Delta x = \hbar/2$ for any α (the average number of quanta $n = |\alpha|^2$ can be arbitrarily great). The coherent states demonstrate most vividly the limiting transition from quantum to classical mechanics.

This approach, however, has a shortcoming: the coherent states can be constructed only for special models. On the other hand, the nodeless states and the states with $p, q \ll n$ are easily constructed via the $1/n$-expansion for an arbitrary potential $V(r)$ and in a number of problems without spherical symmetry (e.g. for the problem of two Coulomb centres [20]). Such states minimize the uncertainty relations as $n \to \infty$, and in this respect are similar to the $|\alpha\rangle$ states. However, there is also a difference: the states $|\alpha(t)\rangle$ are nonstationary, while the states discussed above are stationary because we are not considering the motion of a wave packet along the orbit but averaging over many turns. To consider such motion in detail we must introduce a superposition of states with different energies E_n, which unavoidably results in a rapid smearing of the wave packet for systems with a nonequidistant spectrum. However, such detailed description of the motion of the particle is not required in many cases, at least not for the stationary problems of quantum mechanics.

8. Higher-order terms in the $1/n$-expansion

In conclusion we discuss the asymptotic behaviour of higher-order terms in the $1/n$-expansion. It is well-known that perturbation series in quantum mechanics and field theory are divergent owing to the instability of the vacuum state when the coupling constant changes its sign (Dyson's phenomenon [36]). In the case of the $1/n$-expansion a numerical study of the asymptotic behaviour of $\epsilon^{(k)}$ as $k \to \infty$ was performed.

It was found that

$$\epsilon^{(k)} \sim k! \, a^k k^\beta \qquad \text{as} \quad k \to \infty \tag{8.1}$$

where a and β are computable constants (for their computation 40 to 50 coefficients $\epsilon^{(k)}$ proved to be sufficient). As an example, in figure 9 we give $|a(F)|$ for the $|0, 0, n-1\rangle$ states in the Stark effect problem. As long as F remained smaller than $F_* = 0.2081$, the parameter a was positive. When $F \to F_*$, the parameter $a(F)$ acquired a power singularity. This accounts for the above-mentioned sharp decrease in accuracy in determining the energy via the $1/n$-expansion method at $F \approx F_*$. When $F > F_*$, the parameter $a(F)$ is complex-valued.

Owing to (8.1) the radius of convergence of the $1/n$-expansion (2.3) is zero, and the sum of the series (2.3) has an essential singularity at $1/n = 0$. However, this series can be successfully summed via Padé approximants, which determine both the Stark shift and the level width (in the region $0 < F < F_*$, where $\epsilon^{(k)}(F) < 0$, Padé–Hermite approximants [15, 16] can be used).

Similar results were obtained for the Yukawa and other short-range potentials. In this case it can be shown that

$$a(\nu) \simeq A\left(\frac{\nu_* - \nu}{\nu_*}\right)^{-5/4} \qquad \nu \to \nu_* \tag{8.2}$$

$$A = \frac{5c^3}{96} = 0.1997\left(1 + \frac{x f'''}{3 f''}\bigg|_{x=x_*}\right)^{3/4} \tag{8.3}$$

where c is the same coefficient as in equation (4.18), and the screening function $f(x) \equiv x v(x)$. Therefore, the higher-order terms in the $1/n$-expansion, $\epsilon^{(k)}(\nu)$, grow very rapidly as $\nu \to \nu_*$, which agrees with the results of numerical calculations (the derivation of equation (8.2) and other details will be published elsewhere).

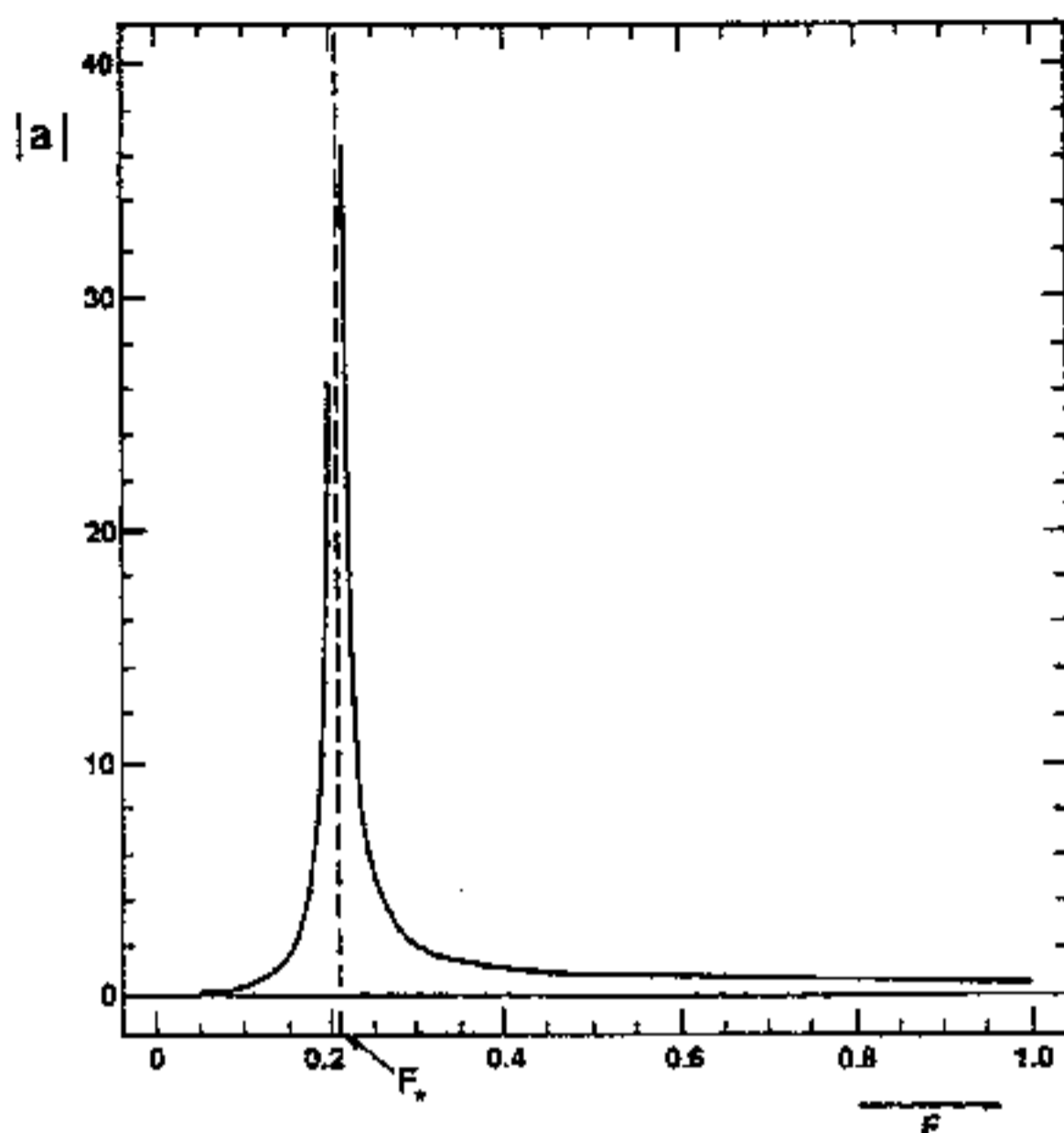**Figure 9.** The dependence of $|a(F)|$ on $F = n^4 \mathcal{E}$, where a is the parameter of asymptotics (8.1).

Acknowledgments

Appendix 1. The 1/n-expansion for wavefunctions

Here we consider in greater detail the $1/n$-expansion for the wavefunctions assuming that $p = 0$ (nodeless states). Employing the method developed in [17], instead of equation (3.4) we obtain

$$\chi_n = \left(\frac{n\omega}{\pi x_0^2}\right)^{1/4} \exp\left(-\frac{\omega\xi^2}{2}\right) \left\{1 + \frac{h_1(\xi)}{\sqrt{n}} + \frac{h_2(\xi)}{n} + O\left(\frac{1}{n^{3/2}}\right)\right\} \tag{A1.1}$$

where $\chi_n \equiv \chi_{n,n-1}(r)$,

$$h_1 = \tfrac{1}{3}a\xi^3 + b\xi \qquad h_2 = \tfrac{1}{6}a_1\xi^6 + \tfrac{1}{4}b_1\xi^4 + \tfrac{1}{2}c_1\xi^2 + d_1 \quad \text{etc} \tag{A1.2}$$

$$a = -A_0^{(1)}x_0^3 \qquad b = -A_1^{(1)}x_0 \qquad a_1 = \tfrac{1}{3}a^2$$
$$b_1 = \left(A_0^{(2)} + \tfrac{4}{3}A_0^{(1)}A_1^{(1)}\right)x_0^4 \qquad c_1 = \left[\left(A_1^{(1)}\right)^2 - A_1^{(2)}\right]x_0^2. \tag{A1.3}$$

Here x_0 and ω are defined by equations (2.2) and (2.6), the $A_j^{(i)}$ are coefficients of the expansion of the logarithmic derivative of the wavefunction and are calculated via the recurrence relations given in [17], and the coefficient d_1 is determined from the normalization condition

$$\int_0^\infty \chi_n^2(r)\,dr = 1 + O(n^{-2}). \tag{A1.4}$$

Expressing the $A_j^{(i)}$ in terms of the v_k introduced by equation (2.7), we finally obtain:

$$a = \frac{\sigma}{2\omega} = \frac{2(1-v_1)}{\omega} \qquad b = \frac{a-1}{\omega} \qquad a_1 = \frac{4(1-v_1)^2}{3\omega^2}$$

$$b_1 = \frac{11a^2 - 8a - c}{6\omega} \qquad c_1 = \frac{2b - 3(1+b_1)}{2\omega} \tag{A1.5}$$

$$d_1 = \frac{24(1+\omega^2 c_1) + 101a^2 - 96a - 3c}{96\omega^3}$$

where $c = 15(1 - v_2)$. These equations make it possible to easily find the first terms in the expansion (A1.1) for an arbitrary potential $V(r)$. The same method can be used to find the next terms in the $1/n$-expansion, but explicit formulae become too cumbersome and it has proved more convenient to employ the recurrence procedure described in [17].

To illustrate the above reasoning, we consider several examples for which comparison with exact solutions is possible.

(a) In the case of the Coulomb potential, $v(x) = x^{-1}$, $v_k = 2(k+3)$, $x_0 = n^2$ and $\omega = 1$, and equations (A1.5) yield

$$a = 1 \qquad b = c_1 = 0 \qquad a_1 = \tfrac{1}{3} \qquad b_1 = -1 \qquad d_1 = -\tfrac{1}{48}. \tag{A1.6}$$

On the other hand, the exact wavefunction of the state with $l = n - 1$ is [26]

$$\chi_n(r) = \frac{2^n r^n}{n^{n+1}\left(\Gamma(2n)\right)^{1/2}} \exp\left(-\frac{r}{n}\right)$$

$$= \frac{\exp(-\xi^2/2)}{(\pi n^3)^{1/4}}\left\{1 + \frac{\xi^3}{3\sqrt{n}} + \left(\frac{\xi^6}{18} - \frac{\xi^4}{4} - \frac{1}{48}\right)\frac{1}{n}\right\} \tag{A1.7}$$

$\left(r = n^2(1 + n^{-1/2}\xi)\right)$, which fully agrees with equation (A1.1) and (A1.6).

(b) For the harmonic oscillator, $v(x) = -x^2/2$, $v_0 = -1/3$, $v_k = 0$ for $k \geq 1$, $x_0 = n^{1/2}$ and $\omega = 2$. Combining all this with (A1.5), we arrive at the same values (A1.6) for all the coefficients, with the exception of $d_1 = 1/48$, which coincides with the expansion of the exact wavefunction for $n \gg 1$:

$$\chi_n(r) = \left(\frac{2}{\Gamma(n+1/2)}\right)^{1/2} r^n \exp\left(-\frac{r^2}{2}\right).$$

(c) In the case of the funnel potential (4.6), all the coefficients in (A1.1) can easily be expressed in terms of the classical oscillation frequency ω:

$$a = \frac{1}{2}\left(\omega + \frac{1}{\omega}\right) \qquad b = \frac{(\omega-1)^2}{2\omega^2} \qquad c = 3(\omega^2 + 2)$$

$$b_1 = -\frac{1}{24\omega^3}\left(\omega^4 + 16\omega^3 + 2\omega^2 + 16\omega - 11\right) \tag{A1.8}$$

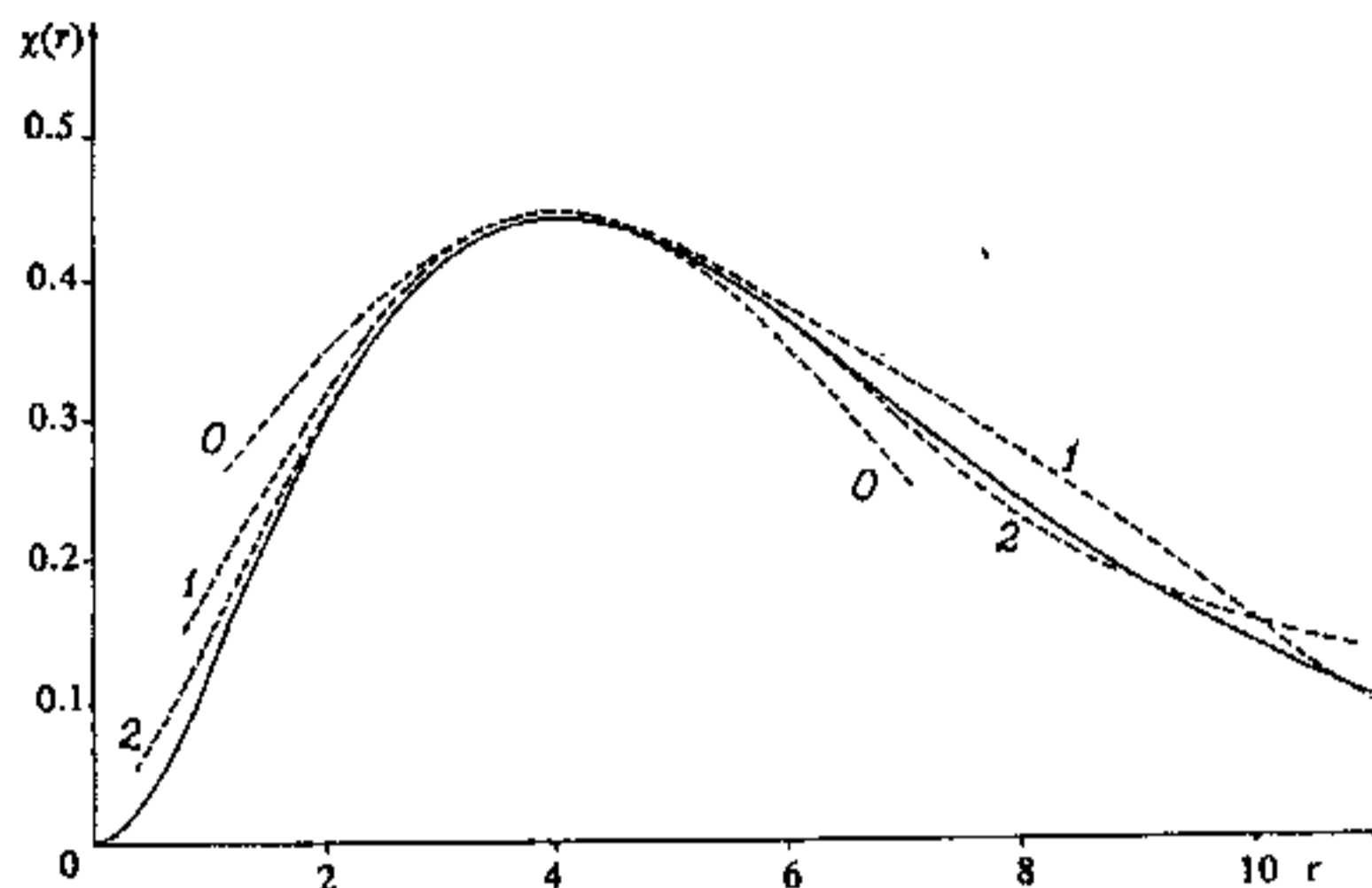$$c_1 = -\frac{1}{16\omega^4}(\omega-1)^2(\omega^2 + 2\omega - 11)$$

Figure A1. The accuracy of the $1/n$-expansion for the wavefunction of the 2p-state ($n = 2$) in the Coulomb potential.

with $\omega = 1 + \nu^2 - \frac{7}{2}\nu^4 + \ldots$ as $\nu \to 0$, and $\omega = \sqrt{3}\,(1 - \frac{1}{3}\nu^{-2/3} + \ldots)$ as $\nu \to \infty$. Note that the case of $\nu \to 0$ corresponds to the Coulomb potential and that of $\nu \to \infty$ to the linear potential, that is, $N = 1$ in equation (4.1).

(d) The Teitz potential [37] serves as a good approximation for the Thomas–Fermi model in neutral atoms. For this potential the screening function $f(x)$ in equation (4.14) is $(1 + x)^{-2}$ and the exact solution of the Schrödinger equation with zero energy is known [38]. For one thing, for nodeless states ($n \geq 2$) the normalized wavefunction is

$$
\begin{aligned}
\chi_n &= \left[\frac{(4n - 3)!}{(2n)!\,(2n - 4)!}\right]^{1/2} \frac{x^n}{(1 + x)^{2n-1}} \\
&= \left(\frac{n \exp(-\xi^2)}{2\pi}\right)^{1/4}\left[1 + \left(\frac{\xi^3}{4} + \frac{\xi}{2}\right)\frac{1}{\sqrt{n}} \right. \\
&\qquad\qquad \left. + \frac{1}{32}(\xi^6 - 3\xi^4 - 37)\frac{1}{n} + \ldots\right].
\end{aligned}
\tag{A1.9}
$$

The dependence of x_0 and $\epsilon^{(0)}$ on ν is determined from the equation

$$
\nu = \frac{x(1 + 3x)}{(1 + x)^3} \qquad \epsilon^{(0)} = \frac{(x - 1)(1 + 3x)}{(1 + x)^6}.
\tag{A1.10}
$$

Thus, $\epsilon^{(0)}(\nu) = 0$ at $x_0 = 1$ where $\nu = \omega = 1/2$, $v_0 = 11/12$, $v_1 = 13/16$, etc, with the result that $a = 3/4$, $a_1 = 3/16$, etc†. In all the cases considered above, the $1/n$-expansion for nodeless wavefunctions has the structure of equation (A1.1).

---

† Note that for states with $E = 0$ the formulae for the coefficients b, b_1, etc have a form somewhat different from (A1.5). The reason is that the coupling constant g in equation (2.1) in this case explicitly depends on n (see equation (2.9)).

Let us now examine the accuracy of this method. We start with the Coulomb potential. In figure A1 the full curve corresponds to the exact wavefunction of the 2p-level and the broken curves 0, 1 and 2 represent three successive approximations to (A1.7), with the label '0' corresponding to a Gaussian packet. We see that already the term proportional to $n^{-1/2}$ considerably improves the agreement of the $1/n$-approximation with the exact solution, and including the term proportional to n^{-1} ensures an accuracy $\lesssim 5\%$ with a sizable range of r values. Similar results have been obtained for the funnel potential [19].

In some cases the Schrödinger equation with zero energy can be solved analytically, say, for the potential

$$V(r) = \tfrac{1}{2}g^2 r^{2\alpha} - \lambda g r^{\alpha-1} \qquad \alpha > 0 \tag{A1.11}$$

which for $\alpha > 1$ is a potential of the Higgs type, at $\alpha = 1$ is reduced to the harmonic oscillator, at $\alpha = 0$ to the Coulomb potential and for $0 < \alpha < 1$ is similar to the funnel potential. The wavefunction with $E = 0$, $p = 0$ and angular momentum $l = n - 1$ has the form†:

$$\chi_n(r) = c_n r^n \exp\left(-\frac{g}{\alpha} r^{\alpha+1}\right)$$

$$c_n = \left(\frac{2g}{\alpha+1}\right)^{(2n+1)/2(\alpha+1)} \left[\frac{\alpha+1}{\Gamma((2n+1)/(\alpha+1))}\right]^{1/2} \tag{A1.12}$$

with $\lambda = n + \alpha/2$ in (A1.11). As $n \to \infty$,

$$\frac{\tilde{c}_n}{c_n} = 1 + \frac{\alpha^2 - 4\alpha + 1}{48(\alpha+1)n} + O\left(\frac{1}{n^2}\right). \tag{A1.13}$$

For $0 \leq \alpha \lesssim 6$ the coefficient of $1/n$ is small numerically, so that formula (3.2) is accurate within one per cent even at $n = 1$. Near the maximum point, $r = r_0 = (n/g)^{1/(\alpha+1)}$, the function $\chi_n(r)$ assumes the form of (A1.1) with $\omega = 1 + \alpha$ and $h_1(\xi) = \tfrac{1}{6}(2 + \alpha - \alpha^2)\xi^3$.

Up to this point we have studied nodeless states, for which the Gaussian function in equation (A1.1) corresponds to the ground state of a one-dimensional oscillator describing zero-point fluctuations in variable ξ. For states with $p \neq 0$ this function is replaced by the wavefunction of the pth excited oscillator state, which leads to a generalization of the above results to states with nodes, provided that $p \ll n$. However, the accuracy of the $1/n$-expansion diminishes as p grows. For instance, for the asymptotic coefficient at zero in (3.1) we get

$$\frac{\tilde{c}_{nl}}{c_{nl}} = 1 + \frac{k_1}{n} + \frac{k_2}{n^2} + \ldots \tag{A1.14}$$

where $n \to \infty$, $p = 0, 1, \ldots$ is fixed and $k_1 = \tfrac{1}{2}(p^2 - \tfrac{1}{24})$ for the harmonic oscillator and $k_1 = \tfrac{1}{8}(7p^2 + 3p + \tfrac{1}{6})$ for the Coulomb potential (these values of k_1 are obtained from comparing equation (3.2) with known exact solutions). We see that the case of $p = 0$ is special because the correction term proportional to $1/n$ is small numerically. It can be verified that the same is true of the subsequent coefficients $k_2, k_3, \ldots$.

† Here if we formally put $n = l + 1 = 0$, we arrive at the one-dimensional problem. For instance, at $\alpha = 3$ from (A1.12) we obtain the zero-energy solution $\psi_0(x) = \text{const} \times \exp(-x^4/4)$ corresponding to the potential $V(x) = (x^6 - 3x^2)/2$.

Appendix 2. On the widths of sub-barrier resonances

Formula (3.4) for the wavefunction $\chi_n \equiv \chi_{n,n-1}(r)$ is valid in the classically allowed region $r_- \leq r \leq r_+$ and can be continued to the sub-barrier region via the WKB method†. To calculate the level width Γ_n we must normalize $\chi_n(r)$ to one particle in the well $r_- < r < r_+$ and find the flux of particles leaving for infinity. Since the quasi-classical momentum $p(r) = nP(r) \propto n$, to evaluate the integral $\int Q \, dr$ in equation (A2.1) we must allow for the correction term proportional to $1/n$ in the energy ϵ (only in this case do we arrive at a formula for Γ_n that is asymptotically exact in the pre-exponential factor). Below we briefly discuss the calculation procedure.

In the sub-barrier region, $P(r) = iQ(r)$, we have

$$\chi_n(r) = \frac{1}{(4\pi e)^{1/4}} \left[\frac{\omega}{r_0^2 Q(r)} \right]^{1/2} \exp\left\{ -n \int\limits_{r_+}^{r} Q(r) \, dr \right\} \tag{A2.1}$$

where

$$Q^2 = \frac{\rho^2}{r^2} - \frac{2v(r)}{\nu} + q^2$$

$$= Q_0^2 - \frac{S_0}{n} + O\left(\frac{1}{n^2}\right)$$

$$Q_0^2 = \frac{1}{r^2} - \frac{2v(r)}{\nu} + q_0^2 \tag{A2.2}$$

$$S_0 = \frac{1}{r^2} + \frac{\omega - 1}{r_0^2} \qquad q^2 = -\frac{\epsilon}{\nu^2}$$

$$q_0^2 = \frac{2v(r_0)}{\nu} - r_0^2 \qquad \rho = \frac{l + 1/2}{n} \qquad 0 < \rho < 1.$$

Using the well-known rules of matching [26], we obtain an outgoing wave corresponding to the quasi-stationary state:

$$\chi_n^{(+)}(r) = \frac{1}{(4\pi e)^{1/4}} \left[\frac{\omega D}{r_0^2 P(r)} \right]^{1/2} \exp\left\{ i\left[\int\limits_{r_2}^{r} P(r) \, dr + \frac{\pi}{4} \right] \right\} \tag{A2.3}$$

where $r > r_2$, and $D = \exp(-2n \int_{r_+}^{r_2} Q(r) \, dr)$ is the barrier penetrability. Employing the expansion $(n \to \infty)$

$$\int\limits_{r_+}^{r} Q(r) \, dr = \int\limits_{r_0}^{r} Q_0(r) \, dr + \frac{1}{n} \int\limits_{r_0}^{r} dr \left[\frac{1}{r - r_0} - \frac{S_0(r)}{Q_0(r)} \right]$$

$$- \frac{1}{4n} \ln\left\{ 4en\omega \left(\frac{r}{r_0} - 1 \right)^2 \right\} + \cdots \tag{A2.4}$$

† *Notation:* r_0 is the minimum point of the effective potential defined by equation (2.2) and $r_\pm$ and r_2 are the turning points, with $r_- < r < r_+$ and $r > r_2$ the classically allowed regions and $r_+ < r < r_2$ the sub-barrier region.

we finally obtain

$$\Gamma_n = 4\pi \left(\frac{en}{\omega}\right)^{1/2} \frac{r_2 - r_0}{r_0} \exp\left\{-2\int_{r_0}^{r_2} dr \left[nQ_0 + \frac{1}{r - r_0} - \frac{S_0}{Q_0}\right]\right\}. \tag{A2.5}$$

To illustrate this general formula, we consider the Stark effect for $|0, 0, n-1\rangle$ states. No 'radial' excitations exist in this case ($n_1 = n_2 = 0$), and with $n \to \infty$ the particle is at rest at the equilibrium point (ξ_0, η_0) determined from equations (5.1). Going over to the scaled variables (5.4), we obtain

$$P(x) = \left(\frac{\epsilon}{4} + \frac{\beta_1}{x} - \frac{\mu^2}{4x^2} - \frac{Fx}{4}\right)^{1/2}$$

$$P(y) = \left(\frac{\epsilon}{4} + \frac{\beta_2}{y} - \frac{\mu^2}{4y^2} + \frac{Fy}{4}\right)^{1/2} \tag{A2.6}$$

$$\psi(r) = \frac{ne^{im\varphi}\chi_n(x)}{n^2\sqrt{xy}} \times \begin{cases} \chi_n(y) & \text{if } y_- < y < y_+ \\ \chi_n^{(+)}(y) & \text{if } y > y_2 \end{cases}$$

where ξ and η are parabolic coordinates ($\xi = n^2 x$, $\eta = n^2 y$) and $\mu = m/n$. The normalization constant N is determined from the condition

$$\frac{\pi n^3}{2} \int |\psi|^2 (x + y)\, dx\, dy = 1 \qquad N = \frac{1}{n}\left[\frac{2x_0 y_0}{\pi(x_0 + y_0)}\right]^{1/2}. \tag{A2.7}$$

Using (A2.6) to calculate the particle flux at infinity, we find the Stark resonance width $\Gamma_n \equiv \Gamma^{(0,0,n-1)}$:

$$\Gamma_n(\mathcal{E}) = \frac{A_n cD}{T_\eta} \tag{A2.8}$$

where

$$D = \exp\left\{-\int_{y_+}^{y_2} \left(\frac{\mu^2}{y^2} - \frac{4\beta_2}{y} - Fy - \epsilon\right)^{1/2} dy\right\} \tag{A2.9}$$

$$A_n = \frac{\sqrt{2}\,\pi}{n!}\left(\frac{n}{e}\right)^{n+1/2} \qquad c = \frac{4y_0}{x_0 + y_0} \qquad T_\eta = \frac{2\pi n^3 y_0^2}{\omega_2}. \tag{A2.10}$$

Note that calculating via the $1/n$-expansion yields $A_n = (\pi/e)^{1/2} = 1.075$. The value of A_n specified in (A2.10) is selected in such a way that equation (A2.8) coincides with the exact asymptotic behaviour of $\Gamma_n(\mathcal{E})$ for all $n = 1, 2, \ldots$ (the numerical values of A_n are very close to unity). The factor c in (A2.8) reflects the fact that the problem is not one-dimensional (at $c = A_n = 1$ equation (A2.8) transforms into the Gamow formula; its generalization to a multidimensional case is given in [39]).

For $n \to \infty$ the turning points y_- and y_+ are close to each other since the wavefunction is concentrated near the classical equilibrium point y_0. Hence,

equations (A2.8) and (A2.9) get simpler and the elliptic integral for the barrier penetrability is expressed in terms of elementary function†:

$$D = (2enz)^{1/2}\left(\frac{2z}{1+z}\right)^2 \exp\left\{-\left[\frac{2n}{3F}(-\epsilon^{(0)})^{3/2}J + \varphi\right]\right\}. \qquad (A2.11)$$

Here

$$z = 2\omega_2 = \frac{\sqrt{1-3\tau}}{1-\tau}$$

$$J = \frac{1}{\sqrt{3}}\left(1-\frac{2z^2}{3}\right)^{-3/2}\left[z - \frac{2z^3}{3} - (1-z^2)\tanh^{-1}z\right] \qquad (A2.12)$$

$$\varphi = \frac{\sqrt{1-3\tau}}{\tau}\left[2 - \sqrt{1+3\tau} - \sqrt{1-3\tau}\right]$$

and τ and $\epsilon^{(0)}$ are determined by equations (5.5) and (5.6). The pre-exponential factor in (A2.8) is also expressed in terms of the variable τ:

$$c = (1+\tau) \qquad T_\eta = \frac{4\pi n^3}{(1-\tau)(1-\tau^2)^2\sqrt{1-3\tau}}. \qquad (A2.13)$$

Substituting these expressions into (A2.8), we arrive at the final formula (5.8). Below we give the expansions of the quantities in this formula in two limiting cases. For $F \to 0$ we have

$$\tau = F + 4F^3 + O(F^P) \qquad x = 1 - \tfrac{1}{2}F - \tfrac{13}{8}F^2 + \ldots$$
$$T_\eta = 4\pi n^3(1 + \tfrac{5}{2}F + \ldots) \qquad\qquad\qquad (A2.14)$$

whence

$$\phi = \tfrac{2}{3}F^{-1} + \ln F + \tfrac{11}{4}F + O(F^2) \qquad p = 1 - \tfrac{9}{2}F + \tfrac{89}{16}F^2 + \ldots. \qquad (A2.15)$$

For $\Gamma_n(\mathcal{E})$ we obtain (5.8), which coincides with the well-known asymptotic formula for level widths of the hydrogen atom in a weak electric field [40, 41]. In the other limiting case, $F \to F_* = 0.2081$, we have

$$\tau = \frac{1}{3}\left(1 - \frac{2^{3/2}}{3}f^{1/2} + \frac{2}{27}f + \ldots\right) \qquad z = \frac{3^{1/2}}{2^{1/4}}f^{1/4} + \ldots \qquad (A2.16)$$

$$\phi = \alpha_0 + \alpha_1 f^{5/4} + \ldots \qquad p = \beta f^{7/8} + \ldots$$

where $\alpha_0 = 1 + 2\ln 2 = 2.386$, $\alpha_1 = 2^{3/4}3^{3/2}/5 = 1.748$, $\beta = (2^{89}3^{-42})^{1/8} = 6.983$ and $f = 1 - F/F_* \to 0$. Here the classical oscillation frequency vanishes:

$$\omega_2 = Cf^{1/4} + \ldots \qquad C = 2^{5/4}3^{1/2} = 0.728 \qquad (A2.17)$$

which is related to loss of stability.

If we allow for (A2.16), we find that the applicability criterion for formula (5.8) has the form (cf [39])

$$f \lesssim n^{-4.5} \qquad (A2.18)$$

which excludes only a narrow region near $F = F_*$ for $n \gg 1$.

---

† The origin of the two terms in (A2.11) is as follows. The first term corresponds to an energy $\epsilon = \epsilon^{(0)}$ equal to the minimum of the effective potential $U_2(\eta)$. The second term $\varphi \propto \epsilon^{(1)}$ allows for the discrepancy between the particle energy and $\epsilon^{(0)}$ owing to zero-point vibrations and the anharmonicity effect.

Appendix 3. The $1/n$-expansion and above-the-barrier Stark resonances

As figure 5 shows, there exists a fairly broad range of electric field strength $\mathcal{E}$ in which the widths of atomic levels depend on $\mathcal{E}$ linearly for all practical purposes†. We immediately note that this refers to the case where the resonance energy E is higher than the top of the potential barrier, that is, $F > F_*$ (in this case the widths Γ_n are already not exponentially small as they are in the weak-field range [40], but the Stark resonances are still isolated and may manifest themselves in experiments [27, 31]). Here we employ the $1/n$-expansion to interpret this fact qualitatively.

Since the ϵ_n'' curves have similar shapes for different (n_1, n_2, m) states [14–16], we limit ourselves to the simplest case of the state $(0, 0, n-1)$ as $n \to \infty$. Here the $1/n$-expansion is reduced to the first term $\epsilon_{\mathrm{cl}} \equiv \epsilon^{(0)}(F)$ coinciding with the energy of a classical particle moving along an equilibrium orbit. If in (5.5) we go over to the variable $u = 1 - \tau^2$, we find that

$$u = 1 - \frac{F^2}{u^8} \qquad \epsilon_{\mathrm{cl}} = 3u^3 - 4u^2 \tag{A3.1}$$

with the stable equilibrium point corresponding to the root $u = u_0(F)$, which becomes equal to unity as $F \to 0$. From (A3.1) follow the perturbation expansions:

$$u_0 = 1 - \sum_{k=1}^{\infty} b_{2k} F^{2k} \qquad \epsilon_{\mathrm{cl}} = -1 - \sum_{k=1}^{\infty} d_{2k} F^{2k} \tag{A3.2}$$

where

$$b_{2k} = \frac{(9k-2)!}{k!\,(8k-1)!} \qquad d_{2k} = 6\,\frac{(9k-4)!}{k!\,(8k-2)!}. \tag{A3.3}$$

We see that the energy $\epsilon_{\mathrm{cl}}(F)$ lowers as the field strength grows and remains real for $0 < F < F_*$ (the perturbation series converge for $F \le F_* = 0.208\,098\ldots$).

In the above-the-barrier region, $F > F_*$, we assume that $u = \rho\mathrm{e}^{-\mathrm{i}t}$, and after elementary but numerous calculations we get

$$\rho = \frac{\sin 8t}{\sin 9t} \qquad F = \frac{(\sin t)^{1/2}(\sin 8t)^4}{(\sin 9t)^{9/2}}$$

$$\epsilon'' = \frac{4\sin 2t\,\sin 3t\,\sin^2 8t}{(\sin 9t)^3}\,(\sin^2 3t - 3\sin^2 t) \tag{A3.4}$$

with $\epsilon'' = -\mathrm{Im}\,\epsilon_{\mathrm{cl}}$ and $0 < t < \pi/9$ ($t = 0$ corresponds to $F = F_*$ and $t \to \pi/9$ to $F \to \infty$). These equations determine in parametric form the dependence of the width of levels with $n \gg 1$ on the electric field strength (we use the reduced variables (5.4)).

Clearly, with $F \to F_*$ we have the expansion (5.7), and with $F \to \infty$,

$$\epsilon_{\mathrm{cl}}(F) = \tfrac{3}{2}\big(1 - \mathrm{i}\sqrt{3}\big)F^{2/3}[1 + O(F^{-2/9})]. \tag{A3.5}$$

† See also [15, 16]. Note that even the very simple approximation, $\epsilon = \epsilon^{(0)} + \epsilon^{(1)}n^{-1}$, ensures reasonable accuracy even for $n \sim 1$, as figure 5 shows. For instance, the error of this approximation is roughly 10% for the level shift and only 3% for the level width at $F = 1$ for the case of the ground state ($1/n = 1$).
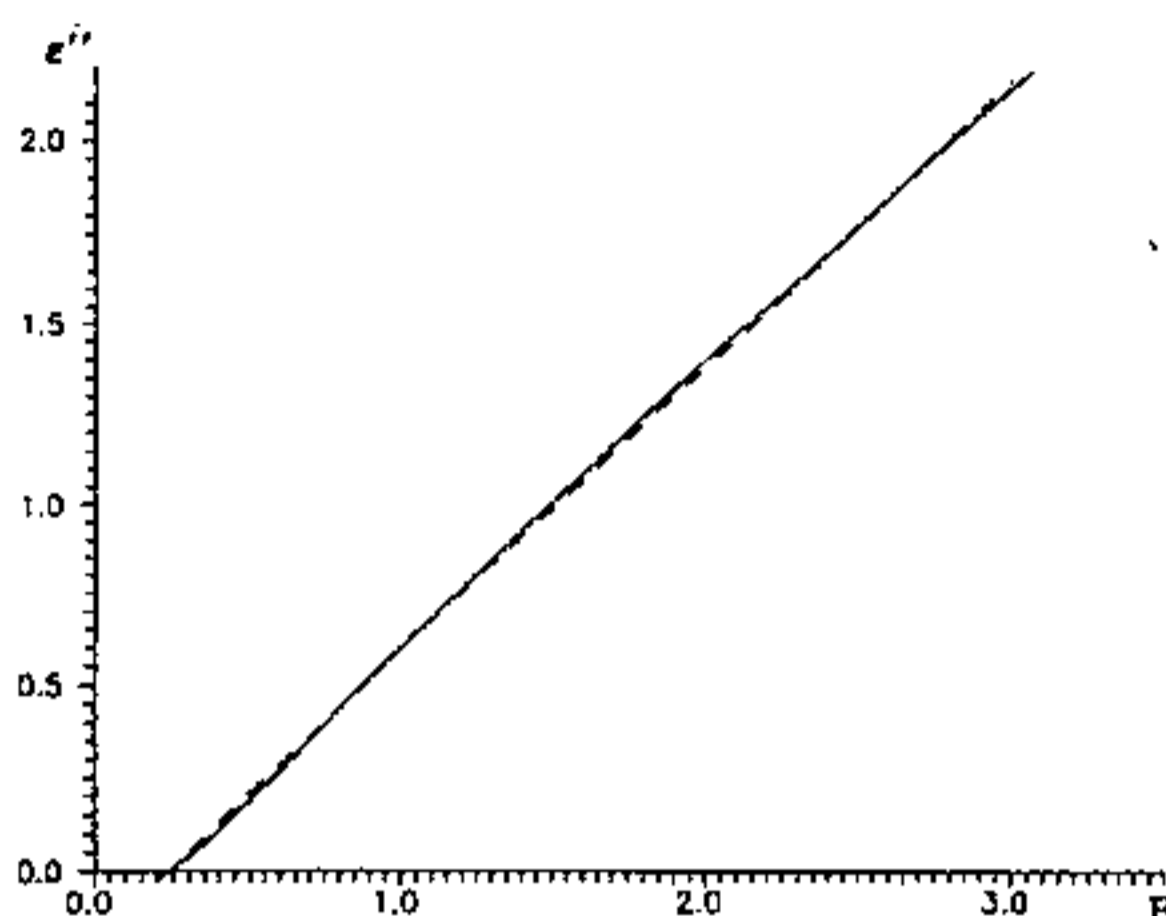
Figure A2. Approximate linearity of the Stark widths ($n \gg 1$). The full curve corresponds to equations (A3.4) and the broken to equation (A3.6).

For an arbitrary F the functions ϵ_{cl} and ϵ'' can easily be calculated via the above equations. This gives the full curve in figure A2, which clearly shows that $\epsilon''(F)$ is practically linear for $F > 0.25$. The broken curve in figure A2 represents the function

$$\tilde{\epsilon}(F) = c(F - F_0) \tag{A3.6}$$

where the parameters c and F_0 were fitted via the least squares method. The results of the fitting procedure are listed in table A1, where we give the parameters c and F_0 (optimal in the sense of σ being minimal) and the mean square deviation σ between the functions $\tilde{\epsilon}$ and ϵ'',

$$\sigma = \left\{ \frac{1}{L} \sum_{i=1}^{L} \left[\epsilon''(F_i) - \tilde{\epsilon}(F_i) \right]^2 \right\}^{1/2} \tag{A3.7}$$

(the fitting interval $F_1 \leq F \leq F_L$ and the number of points, L, were varied). From table A1 it follows that the widths Γ_n in the above-the-barrier regions are almost linear functions of field strength. The same follows from figure A2, in which the worst

Table A1. The optimal parameters in equation (A3.6).

No	c	F_0	L	F_1	F_L	$\sigma \times 100$
1	0.785	0.234	105	0.2081	3.0	1.34
2	0.802	0.2526	48	0.25	1.5	0.72
3	0.803	0.2529	51	0.25	2.0	0.70
4	0.8098	0.2600	40	0.30	2.0	0.416
5	0.8101	0.2601	80	0.30	2.0	0.415
6	0.8104	0.2601	399	0.30	2.0	0.413

approximation (No 1) is depicted by a broken curve (for this approximation $F = F_*$, and for $F \approx D_*$ the function $\epsilon''(F)$ is sure to be nonlinear). For the variants 2–6 we took $F > 1.2F_*$. As a result the $\epsilon''(F)$ and $\tilde{\epsilon}(F)$ curves coincide within the accuracy of the diagram.

But what properties of the interaction potential are related to the 'linearity' of the $\Gamma_n(\mathcal{E})$ widths Let us consider a model problem with the potential

$$V(r) = -\frac{1}{r} - g\,\frac{r^N}{N} \qquad N > 0 \tag{A3.8}$$

which for $N = 1$ and $g < 0$ transforms into the funnel potential (4.6) and for $g > 0$ is the spherical analogue of the Stark problem for the hydrogen atom. Applying the $1/n$-expansion to states with $l = n - 1 \gg 1$ yields the following equations:

$$x - x^{N+2} = \nu \qquad \epsilon_{\rm cl} = -\left(1 - x_0^{N+1}\right)\left(1 + \frac{N+2}{N}\,x_0^{N+1}\right) \tag{A3.9}$$

with the first defining the classical equilibrium point $x_0(\nu)$; here $\nu = n^2 g^{1/(N+1)}$ and $\hbar = m = 1$. The solution to equations (A3.9) is real as long as $\nu < \nu_*$,

$$\nu_* = (N + 1)(N + 2)^{-(N+2)/(N+1)} \tag{A3.10}$$

and enters the complex plane when $\nu > \nu_*$. For the imaginary part of $\epsilon_{\rm cl}(\nu)$ equations can be derived that are similar to (A3.4) and imply that only at $N = 1$ is the dependence of ϵ'' on the effective coupling constant†, $\lambda = \nu^{N+1} = n^{2N+2}g$, linear, with a good accuracy, in the above-the-barrier region. Since $\epsilon'' \propto (\lambda - \lambda_*)^{3/2}$ for $\lambda \to \lambda_* = 4/27$ and $\epsilon'' \propto \lambda^{2/3}$ for $\lambda \to \infty$, the dependence (A3.6) can be called the 'intermediate asymptotics'. Thus, the 'linearity' of $\epsilon''(F)$ in the above-the-barrier region is specific at $N = 1$, that is, is due to the special shape of the potential, $-\mathcal{E}z$, acting on the atom. (Note that for $F \gtrsim F_*$ there is no way in which this potential can be considered a perturbation of the Coulomb potential.)

References

[1] Ferrel R A and Scalapino D J 1974 *Phys. Rev.* A **9** 846
[2] Witten E 1979 *Nucl. Phys.* B **149** 285
      Witten E 1980 *Phys. Today* **33** 38
[3] Dolgov A D and Popov V S 1979 *Phys. Lett.* **86B** 185
[4] Dolgov A D, Eletskiĭ V L and Popov V S 1979 *Preprint ITEP No. 72* (Moscow: Institute of Theoretical and Experimental Physics)
[5] Mlodinow L D and Papanicolaou N 1980 *Ann. Phys., NY* **128** 314
      Mlodinow L D and Papanicolaou N 1981 *Ann. Phys., NY* **131** 1
[6] Bender C M, Mlodinow L D and Papanicolaou N 1982 *Phys. Rev.* A **25** 1305
[7] Imbo T, Pagnamenta A and Sukhatme U 1984 *Phys. Rev.* D **29** 1669
      Imbo T, Pagnamenta A and Sukhatme U 1984 *Phys. Lett.* **105A** 183
[8] Yaffe L G 1982 *Rev. Mod. Phys.* **54** 407
[9] Yaffe L G 1983 *Phys. Today* **36** 50
[10] Chatterjee A 1990 *Phys. Rep.* **186** 249
[11] Popov V S, Vaĭnberg V M and Mur V D 1985 *JETP Lett.* **41** 539

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† At $N = 1$ the coupling constant $\lambda = n^4 g$ is analogous to the reduced electric field $F = n^4 \mathcal{E}$.

[12] Popov V S, Vaĭnberg V M and Mur V D 1986 *Yad. Fiz.* **44** 1103
[13] Popov V S and Vaĭnberg V M 1986 *Sov. Phys. Dokl.* **31** 620
   Popov V S, Vaĭnberg V M and Mur V D and Shcheblykin A V 1987 *Sov. Phys. Dokl.* **32** 295
   Mur V D, Pozdnyakov S G and Popov V S 1988 *Sov. Phys. Dokl.* **33** 272
[14] Popov V S, Mur V D, Shcheblykin A V and Vaĭnberg V M 1987 *Phys. Lett.* **124A** 77
[15] Vaĭnberg V M, Mur V D, Popov V S and Sergeev A V 1986 *JETP Lett.* **44** 9
   Vaĭnberg V M, Mur V D, Popov V S and Sergeev A V 1987 *Sov. Phys.-JETP* **66** 258
[16] Popov V S, Mur V D, Sergeev A V and Vaĭnberg V M 1990 *Phys. Lett.* **149A** 418, 425
[17] Vaĭnberg V M, Mur V D, Popov V S, Sergeev A V and Shcheblykin A V 1988 *Teor. Mat. Fiz.* **74** 399
[18] Mur V D and Popov V S 1987 *JETP Lett.* **45** 410
   Mur V D and Popov V S 1988 *Yad. Fiz.* **47** 697
[19] Mur V D, Pozdnyakov S G and Popov V S 1990 *Yad. Fiz.* **51** 390
[20] Mur V D, Popov V S and Sergeev A V 1990 *Sov. Phys.-JETP* **70** 16
[21] Mur V D and Popov V S 1990 *Sov. Phys.-JETP* **70** 975
[22] Vaĭnberg V M, Popov V S and Sergeev A V 1990 *Sov. Phys.-JETP* **71** 470
[23] Eichten E, Gottfried K, Kinoshita T, Lane K D and Yan T M 1978 *Phys. Rev.* D **17** 3090
   Eichten E, Gottfried K, Kinoshita T, Lane K D and Yan T M 1980 *Phys. Rev.* D **21** 203, 313
[24] Badalyan A M, Ioffe B L and Smilga A V 1987 *Nucl. Phys.* B **281** 85
[25] Badalyan A M, Kitoroage D I and Pariysky D S 1987 *Yad. Fiz.* **46** 226
[26] Landau L D and Lifshitz E M 1977 *Quantum Mechanics: Non-relativistic Theory* 3rd edn (Oxford: Pergamon)
[27] Mur V D and Popov V S 1988 *JETP Lett.* **48** 70
   Mur V D and Popov V S 1988 *Sov. Phys.-JETP* **67** 2027
[28] Franceschini V, Grecchi V and Silverstone H J 1985 *Phys. Rev.* A **32** 1338
[29] Benassi L and Grecchi V 1980 *J. Phys. B: At. Mol. Phys.* **13** 911
[30] Damburg R J and Kolosov V V 1976 *J. Phys. B: At. Mol. Phys.* **9** 3149
   Damburg R J and Kolosov V V 1978 *J. Phys. B: At. Mol. Phys.* **11** 1921
[31] Kolosov V V 1987 *J. Phys. B: At. Mol. Phys.* **20** 2359
   Kolosov V V 1989 *J. Phys. B: At. Mol. Opt. Phys.* **22** 833
[32] Telnov D A 1989 *J. Phys. B: At. Mol. Opt. Phys.* **22** L399
[33] Alvarez G and Silverstone H J 1989 *Phys. Rev. Lett.* **63** 1364
[34] Alvarez G, Damburg R J and Silverstone H J 1991 *Phys. Rev.* A **44** 3060
[35] Glauber R J 1963 *Phys. Rev. Lett.* **10** 84
   Glauber R J 1963 *Phys. Rev.* A **131** 2766
[36] Dyson F J 1952 *Phys. Rev.* **85** 631
[37] Tietz T 1956 *J. Chem. Phys.* **25** 787
[38] Demkov Yu N and Ostrovskiĭ V N 1972 *Sov. Phys.-JETP* **35** 66
[39] Popov V S, Mur V D and Sergeev A V 1991 *Phys. Lett.* **157A** 185
   Mur V D, Popov V S and Sergeev A V 1991 *Yad. Fiz.* **54** 950
[40] Yamabe T, Tachibana A and Silverstone H J 1977 *Phys. Rev.* A **16** 877
[41] Damburg R J and Kolosov V V 1979 *J. Phys. B: At. Mol. Phys.* **12** 2637