Theory of the Stark effect in a strong field: critical fields, above-barrier resonances, dependence on dimensionality

V. S. Popov

International Institute of Physics, 125040 Moscow, Russia and Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia

V. D. Mur

Moscow Engineering Physics Institute, 115409 Moscow, Russia

A. V. Sergeev

S. I. Vavilov State Institute of Optics, 199034 St. Petersburg, Russia
(Submitted 20 April 1994; resubmitted 28 June 1994)
Zh. Eksp. Teor. Fiz. **106**, 1001–1030 (October 1994)

We have calculated accurate values of the critical fields $\mathscr{E}_c$ for various states (n_1,n_2,m) of a hydrogen atom, including the ground state. Using modified quantization rules that take into account the finite permeability of the barrier, we determine how the widths Γ_n of the Stark resonances depend on n for $\mathscr{E}=\mathscr{E}_c$. Using the $1/n$ expansion, along with several model problems as examples, we show that in the above-barrier $(\mathscr{E}>\mathscr{E}_c)$ region the Stark widths $\Gamma_n(\mathscr{E})$ depend almost linearly on the electric field. We determine the dependence of the ground-state Stark shift of a D-dimensional "hydrogen atom" on the dimensionality D of the space. © *1994 American Institute of Physics.*

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Recently there has been considerable interest in the study of near-threshold resonances in scattering by a potential barrier (i.e., the case $E_r \approx U_m$, where $E_r=\mathrm{Re}\,E$ gives the position of a resonance $E=E_r-i\Gamma/2$ and U_m is the barrier height).[1–9] The problems that arise in this case can be partially solved using semiclassical quantization rules that take into account the finite permeability of the barrier (see Refs. 10, 11 and the citations mentioned therein, and also Ref. 12). In this article we will address the following problems:

1. Accurate calculation of the critical field $\mathscr{E}_c(n_1,n_2,m)$ at which a hydrogenic level with parabolic quantum numbers n_1, n_2, m "touches" the top of the barrier, i.e., $E_r=U_m$. The quantity $\mathscr{E}_c$ separates two characteristic energy regions: a "weak-field" region, in which the levels are below the barrier and have widths $\Gamma_n(\mathscr{E})$ that are exponentially small (and that approach the semiclassical asymptotic limit[13–15] as $\mathscr{E}\to 0$), and a "strong-field" region $(\mathscr{E}>\mathscr{E}_c)$, in which the $\Gamma_n(\mathscr{E})$ increase linearly with $\mathscr{E}$ (see (6.1) below). Both experimental data and numerical calculations show that the transition from one region to the other is abrupt. Therefore, the critical fields $\mathscr{E}_c$, which are rigorously defined only in the semiclassical limit $n\gg 1$, remain meaningful even for low-lying quantum levels such that $n=n_1+n_2+|m|+1$.

2. In Secs. 3 and 4 we discuss the Rydberg limit $n\to\infty$, in which the critical field $F_c=n^4\mathscr{E}_c\to F_*$, where F_* is the classical ionization threshold.[9,16] For states with magnetic quantum number $m=0$, which are often encountered in practice, F_* can be computed analytically. The dependence of F_* on the electric quantum number $k=n_1-n_2$ has been analyzed in detail.[17]

3. It is interesting to study the position and, more especially, the widths of the Stark resonances for $\mathscr{E}=\mathscr{E}_c$. In Sec. 5 we show that the widths $\Gamma_n(\mathscr{E}_c)$ decrease rapidly with increasing n, so that resonances have small widths even when they "touch" the top of the barrier. We estimate Γ_n for the case where the potential that binds the particle is not necessarily Coulombic at small distances, but has a more general power-law form $V(r)\propto r^{-\alpha}$ as $r\to 0$.

4. The subject of Sec. 6 is the Stark resonances in the above-barrier region $\mathscr{E}>\mathscr{E}_c$. In this section we explain the regime in which $\Gamma_n(\mathscr{E})$ is linear with $\mathscr{E}$ using the semiclassical $1/n$ expansion,[18–22]. Discussion of model examples shows that the linearity of $\Gamma_n(\mathscr{E})$ for $\mathscr{E}>\mathscr{E}_c$ is connected with the form of the external potential, i.e., $\mathscr{E}z$, that acts on an electron in the atom.

5. Finally, in Sec. 7 we address the question of how the Stark shift depends on the dimension of the space D in which the atom resides, which is interesting with regard to the method of dimensional scaling[20] widely used at the present time, as well as the $1/n$ expansion.

Some the results of this paper were reported in Refs. 23, 24. Computational details and a discussion of the one-dimensional problem are left to the Appendices.

2. CRITICAL FIELDS IN THE HYDROGEN ATOM

In the problem of the Stark effect in a hydrogen atom, the variables separate in parabolic coordinates ξ, η, φ. The barrier depends only on the variable $\eta=r-z$; for this coordinate, the effective energies E_{eff} and potential equal[13]

$$E_{\mathrm{eff}}=\frac{1}{4}E,\quad U_2(\eta)=\frac{m^2-1}{8\eta^2}-\frac{\beta_2}{2\eta}-\frac{1}{8}\mathscr{E}\eta. \tag{2.1}$$

Since the wave function $\chi_2(\eta)$ is defined on the half-axis $0<\eta<\infty$, $\eta=0$ is a singular point. The Langer transformation[25–27]

 1063-7761/94/100547-14$10.00 © 1994 American Institute of Physics

$$\eta=\exp(u), \quad \chi_2(\eta)=\exp(u/2)\psi(u), \tag{2.2}$$

which places this singularity at $-\infty$, allows us to correctly include the boundary condition $\chi_2(0)=0$ in the semiclassical approach. The resulting Schrödinger equation has the form

$$\frac{1}{n^2}\frac{d^2\psi}{du^2}+p^2\psi=0, \quad p^2=\frac{2\eta^2}{n^2}\left[\frac{1}{4}E-U_2(\eta)\right]-\frac{1}{4n^2}$$

$$=-\frac{\mu^2}{4}+\beta_2 y+\frac{1}{4}\epsilon y^2+\frac{1}{4}Fy^3, \tag{2.3}$$

where $y=n^{-2}\eta$ and β_2 is the separation constant. We use atomic units ($\hbar=e=m_e=1$) and the "reduced" variables

$$\epsilon=\epsilon'-i\epsilon''=2n^2E_n(\mathscr{E}), \quad \epsilon''=n^2\Gamma_n, \quad F=n^4\mathscr{E},$$
$$\mu=\frac{|m|}{n}, \quad \nu_i=\frac{n_i+1/2}{n}, \quad \mu+\nu_1+\nu_2=1, \tag{2.4}$$

$E=E_r-i\Gamma/2$ is the complex resonance energy, n_1,n_2,m are the parabolic quantum numbers, and $n=n_1+n_2+|m|+1$ is the principal quantum number of the level. In going from Eq. (2.1) to (2.3), we have replaced the coefficient m^2-1 by m^2, which corresponds to including the Langer correction.

The condition that the level "touch" the top of the barrier (i.e., $E_r=U_m$) is expressed by the equations

$$\mathrm{Re}\ p^2(y_m)=\mathrm{Re}\left.\frac{dp^2}{dy}\right|_{y=y_m}=0, \tag{2.5}$$

which, taking into account (2.3), have the form

$$\epsilon'=\mu^2y^{-2}-4\beta_2'y^{-1}-Fy, \quad \beta_2'+\tfrac{1}{2}\epsilon'y+\tfrac{3}{4}Fy^2=0. \tag{2.6}$$

The first of these equations can be simplified, while the second can be solved in explicit form[1)]:

$$\epsilon'+2Fy+\mu^2y^{-2}=0, \quad y=-\frac{\epsilon'}{3F}[1+(1-\zeta)^{1/2}], \tag{2.7}$$

where $y=y_m$ is the point at which the potential is maximum, $\zeta=12\beta_2'F/\epsilon'^2$, and $\epsilon'<0$. Substituting the second of the Eqs. (2.7) into the first equation and letting $w=2\sqrt{1-\zeta}$, we find that

$$w^3+3w^2=4(1-\lambda), \tag{2.8}$$

where $\lambda=27\mu^2F^2/(-\epsilon')^3$. From this it follows, in particular, that for an arbitrary state $0\leqslant\lambda<1$ (and thus $3/4\leqslant\zeta<1$). The calculation of the critical field F_c thereby reduces to solving the cubic Eq. (2.8). The quantities ϵ' and β_2' entering into this cubic depend on the reduced electric field F and the quantum numbers of the state. Once they are calculated by summing perturbation-theory terms using Padé–Hermite approximants,[9,28] the critical field $F_c(n_1,n_2,m)$ can be determined from (2.7) or (2.8).

We note that for the states $(n_1,n_2,0)$ Eq. (2.7) is satisfied for

$$y_m=-\frac{\epsilon'}{2F}=2\left(\frac{\beta_2'}{F}\right)^{1/2}, \quad \epsilon'=-4(\beta_2'F)^{1/2}, \quad \zeta=\frac{3}{4}, \tag{2.9}$$

which agrees with Eqs. (16) from Ref. 9 as $n\to\infty$, which in turn are consequences of the Bohr–Sommerfeld quantization rules. In this case, $\zeta=(3/4)z_2$, where $z_2=1$ is the singularity of the hypergeometric function $F(1/4,3/4;2;z_2)$ contained in these equations that is closest to zero [see also (3.1)].

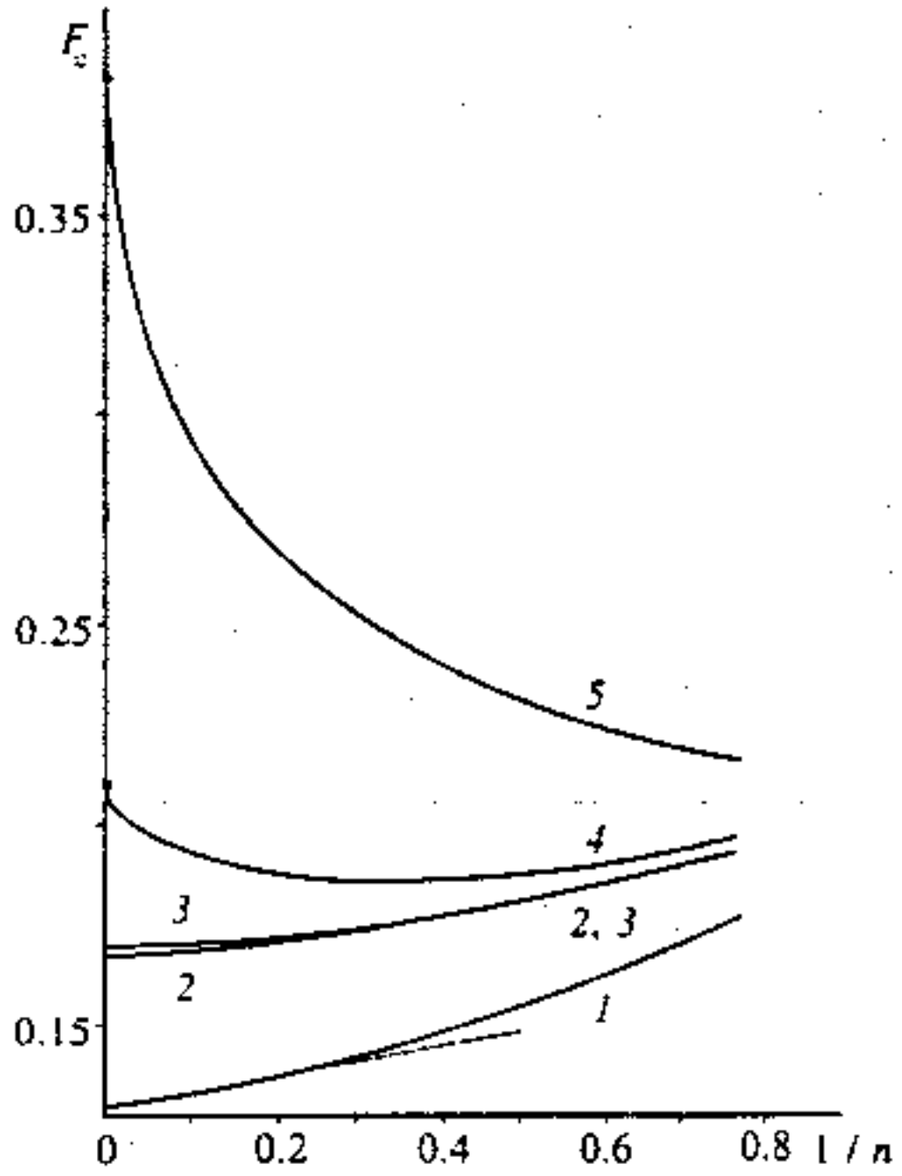

FIG. 1. Critical fields $F_c(n_1,n_2,m)$ in the hydrogen atom. Curves *1–5* correspond to the following series of states: $(0,n-1,0)$, $(n_1;n_1,0)$, (m,m,m), $(0,0,n-1)$, and $(n-1,0,0)$, where $n_1=(n-1)/2$ and $m=(n-1)/3$, n is the principal quantum number of the level. The dashed curve was plotted using the asymptotic Eq. (2.11) for the states $(0,n-1,0)$.

Another limiting case is that of the Rydberg states ($n\gg1$) labeled $(0,0,n-1)$, which correspond to circular orbits of an electron and are closest to classical mechanics. It is not difficult to verify that Eqs. (2.7), in the limit $n\to\infty$ and $\mu=1$, have the (real) solution

$$F_c=2^{12}\cdot3^{-9}=0.2081, \quad \epsilon_c=-2^8\cdot3^{-5}=-1.0535,$$
$$\beta_2=\tfrac{4}{9}, \tag{2.10}$$

where here $y_m=27/16$ and $\zeta=1$.

The results of calculating $F_c=n^4\mathscr{E}_c$ for the various states (n_1,n_2,m) of a hydrogen atom are shown in Fig. 1. We note that the Langer correction is always present for small n, especially in the case of the ground state (see Ref. 23 for more detail). As n increases, the values of F_c approach the classi-

TABLE I.

State	κ	μ	F_*	$-\epsilon_*$	β_2	y_m
$(0,n-1,0)$	-1	0	0.1298	1.441	1.000	5.552
$(n_1,n_1,0)$	0	0	0.1674	1.076	0.432	3.213
(m,m,m)	0	1/3	0.1693	1.073	0.434	3.135
$(0,0,n-1)$	0	1	0.2081	1.053	0.444	1.687
$(n-1,0,0)$	1	0	0.3834	0.000	0.000	0.000

Note. the numbers in this table refer to the Rydberg limit $n\to\infty$: F_* is the classical ionization threshold, ϵ_* is the corresponding reduced energy, etc. In this case the parameters ν_1 and ν_2 from (2.4) equal $\nu_{1,2}=(1-\mu\pm>\kappa)/2$.

TABLE II. Parameters of the Rydberg States $(n_1,n_2,0)$ at the "critical point".

κ	z	F_*	$-\epsilon_*$	y_m	γ	l_0
−1.0	0.000	0.1298	1.441	5.552	0.2120	2.286
−0.75	0.17417	0.1368	1.365	4.991	0.2203	2.312
−0.5	0.41421	0.1450	1.281	4.416	0.2297	2.341
−0.25	0.76386	0.1550	1.186	3.825	0.2403	2.372
0.	1.3147	0.1674	1.076	3.213	0.2523	2.405
0.25	2.2933	0.1837	0.945	2.571	0.2658	2.437
0.5	4.4362	0.2068	0.780	1.886	0.2803	2.458
0.6	6.1778	0.2193	0.699	1.594	0.2857	2.455
0.7	9.2923	0.2351	0.605	1.286	0.2898	2.434
0.75	1.195(1)	0.2448	0.550	1.123	0.2907	2.410
0.80	1.617(1)	0.2564	0.489	0.953	0.2901	2.370
0.85	2.375(1)	0.2705	0.418	0.773	0.2866	2.298
0.90	4.058(1)	0.2888	0.333	0.577	0.2771	2.160
0.95	1.008(2)	0.3153	0.223	0.353	0.2516	1.829
1.00	∞	0.3834	0.000	0.000	-	-

Note. $\kappa=(n_1-n_2)/n$, the magnetic quantum number $m=0$; the values of F_* and ϵ_* are given in atomic units; the power of ten of the numbers is shown in parentheses, i.e., $a(n)\equiv a\cdot 10^n$.

cal ionization threshold F_*, which depends only on the ratios ν_i.[16] Numerical values of F_* are given in Table I and in Table II (for states with magnetic quantum number $m=0$). In the latter case, F_* can be calculated analytically (see the next section). In Tables I and II we also show values of other physical quantities (reduced energy ϵ, separation constant β_2, etc.) that pertain to $F=F_*$.

It can be shown that for the Rydberg states,

$$\frac{F_c}{F_*}=1+\frac{c_1}{n}+\frac{c_2}{n(\ln n+c_3)}+\ldots, \quad n\gg 1, \tag{2.11}$$

where[2)] the c_k are coefficients that depend on ν_1, ν_2, and μ. As an example, for the states $(0,n-1,0)$ we have

$$F_*=\frac{2^{10}}{(3\pi)^4}, \quad c_1=2-2^{5/2}\pi^{-1}, \quad c_2=\tfrac{1}{2}\ln 2,$$

$$c_3=C+\ln 96\pi-4, \tag{2.12}$$

where $C=0.577...$—is Euler's constant (see also Table III). We relegate the details of the derivation of Eq. (2.11) to

TABLE III. Coefficients c_k (numerical values).

k	c_k	k	c_k
1	0.199	12	1.869
2	0.347	13	0.9034
3	2.286	14	7.455(−2)
4	1.200	15	1.915(−2)
5	1.574	16	6.97(−3)
6	0.685	17	3.349(−2)
7	6.002(−1)	18	1.613(1)
8	1.756	19	1.495(−1)
9	1.994(−1)	20	0.100
10	2.004(−1)	21	1.476
11	2.069	22	5.159

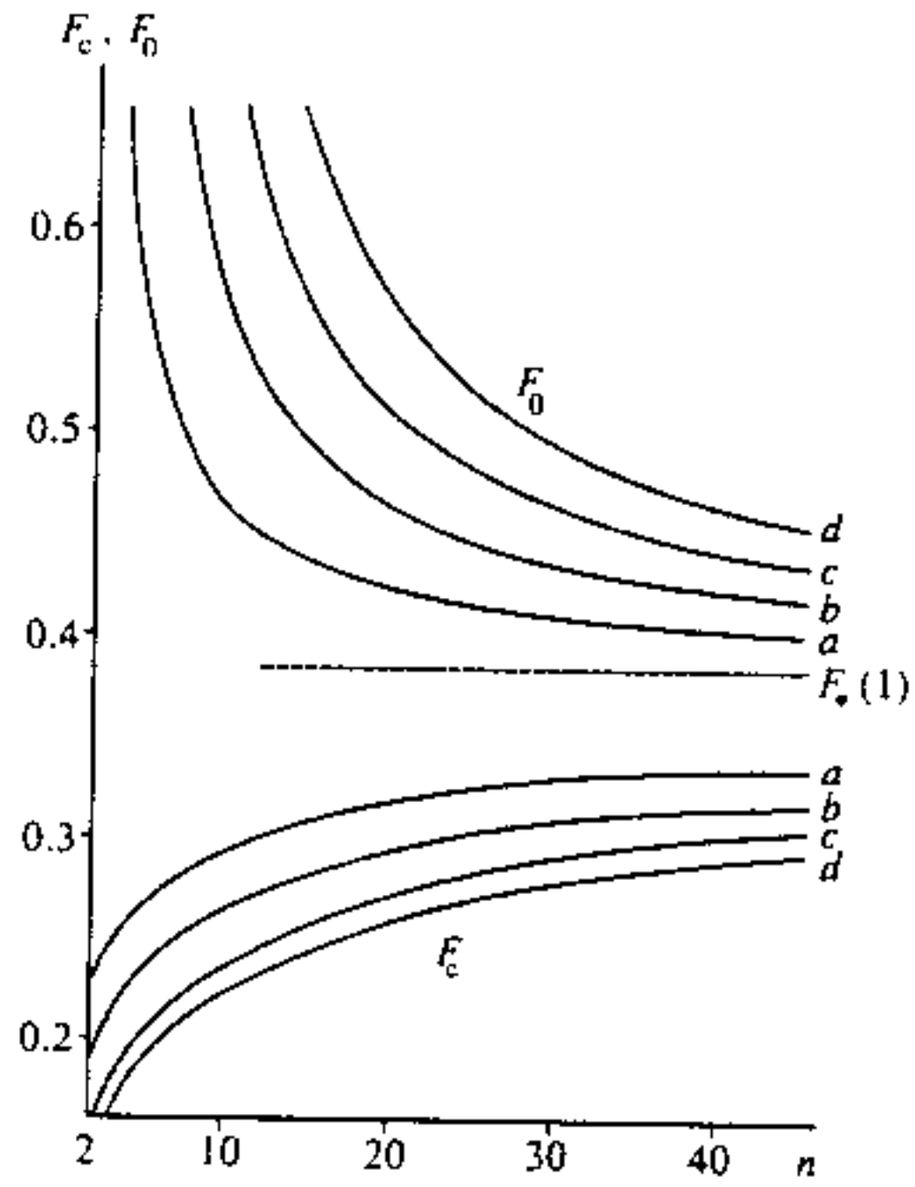

FIG. 2. The fields F_c and F_0 for the states $(n-1,0,0)$ and states with nearby quantum numbers. The curves are denoted a, b, c, and d for states $(n-1,0,0)$, $(n-2,0,1)$, $(n-2,1,0)$, and $(n-3,1,1)$. Here F_c is the critical field $(E_r=U_m)$ and F_0 is the ionization limit (i.e., the field for which $E_r=0$).

Appendix A. As is clear from Fig. 1, this asymptotic form is in good agreement with numerical calculations even at $n\sim 3$.

In Fig. 2 we show the results of calculations of the critical fields F_c for the states $(n-1,0,0)$ and those states close to them that are of experimental interest.[2,3]

The curves of Fig. 1 all terminate on the vertical axis at points corresponding to F_*. For $m=0$ the equation that determines F_* can be written in analytic form. Let us address this question now.

3. CALCULATION OF THE CLASSICAL IONIZATION THRESHOLD F_*

As $n\to\infty$ (the "Rydberg limit"), the quantization condition for $m=0$ takes the form[3)]

$$\beta_1(-\epsilon)^{-1/2}F(1/4,3/4;2;-z_1)=\nu_1,$$

$$\beta_2(-\epsilon)^{-1/2}F(1/4,3/4;2;z_2)=\nu_2, \quad \beta_1+\beta_2=1, \tag{3.1}$$

where $z_i=16\beta_i F/\epsilon^2$, and F is the reduced electric field, while $F(\ldots;z)={}_2F_1(\ldots;z)$ is the Gauss hypergeometric function. The solution to these equations remains real until z_2 reaches unity (i.e., the singular point for the hypergeometric function). This determines the classical ionization threshold $F_*(\kappa)$. As $F\to F_*$ a singularity appears in the solution to the system (3.1), while for $F>F_*$ the solution migrates into the complex plane, corresponding to an above-barrier resonance. In this case Eq. (3.1) should be analytically continued in accordance with the prescription given in Ref. 29.

Here we limit ourselves to calculating the classical ionization threshold F_* and the values ϵ_* and β_i^* corresponding to it. Setting $z_2=1$ in (3.1), and writing $z=z_1/z_2$ and $\kappa=(n_1-n_2)/n$ so that

$$\nu_1=\frac{1+\kappa}{2},\quad \nu_2=\frac{1-\kappa}{2},\quad -1<\kappa<1, \tag{3.2}$$

we obtain the following expression for determining $z=z(\kappa)$:

$$f(z)=c_4\frac{1+\kappa}{1-\kappa}, \tag{3.3}$$

in which $c_4=2^{7/2}/3\pi$, and

$$f(z)=zF(1/4,3/4;2;-z)$$

$$\equiv z(1+z)^{-1/4}F\left(1/4,5/4;2;\frac{z}{1+z}\right). \tag{3.4}$$

We note that

$$f(z)=\begin{cases} z-\dfrac{3}{32}z^2+\ldots, & z\to 0,\\ c_5 z^{3/4}[1-c_6 z^{-1/2}+O(z^{-1})], & z\to\infty, \end{cases}$$

where $c_5=8/3(K/\pi)^{1/2}$, $c_6=3/4K$, and K is a numerical constant which we will often encounter in what follows:

$$K=(1/8)\{\Gamma(1/4)/\Gamma(3/4)\}^2\approx 1.094\ 219\ldots. \tag{3.5}$$

Equation (3.3) has a (unique) solution for all κ which increases monotonically with κ and has a singularity for $\kappa=1$:

$$z(\kappa)=\begin{cases} c_7(1+\kappa)+O((1+\kappa)^2), & \kappa\to -1,\\ c_8(1-\kappa)^{-4/3}+O((1-\kappa)^{-2/3}), & \kappa\to 1, \end{cases} \tag{3.6}$$

where $c_7=c_4/2$ and $c_8=4(\pi K)^{-2/3}$. The remaining quantities at the "critical point" can be computed from the expressions

$$F_*(\kappa)=\frac{2^{14}}{(3\pi)^4}(1-\kappa)^{-4}(1+z)^{-3},$$

$$\epsilon_*=-\frac{2^9}{(3\pi)^2}[(1-\kappa)(1+z)]^{-2},$$

$$y_m=\left[\frac{3\pi}{8}(1-\kappa)\right]^2(1+z),$$

$$\beta_1^*=\frac{z}{1+z},\quad \beta_2^*=(1+z)^{-1} \tag{3.7}$$

(in this case $\zeta_*=12\beta_2^*F_*/\epsilon_*^2=3/4$, independent of the value of κ).

Let us consider the limiting cases.

a) As $\kappa\to -1$ we have

$$F_*(\kappa)=F_*(-1)\{1+c_9(1+\kappa)+\ldots\},$$

$$\epsilon_*(\kappa)=\epsilon_*(-1)\{1-c_{10}(1+\kappa)+\ldots\}, \tag{3.8}$$

where $1+\kappa=(2n_1+1)/n$, $c_9=2(1-3/4c_4)$, $c_{10}=c_4-1$; the values of $F_*(-1)$ and $\epsilon_*(-1)$ are given in Table II.

b) As $\kappa\to 1$ the quantities under discussion have power-law singularities:

$$F_*(\kappa)=F_*(1)[1-c_{11}(1-\kappa)^{2/3}+\ldots],$$

$$\epsilon_*(\kappa)=-c_{12}(1-\kappa)^{2/3}+\ldots, \tag{3.9}$$

where

$$F_*(1)=\left(\frac{16}{9\pi}K\right)^2=0.3834,\quad \epsilon_*(1)=0,$$

$$c_{11}=\left(\frac{8}{c_{12}}\right)^{1/2}=\frac{3}{2}\left(\frac{K^2}{\pi}\right)^{-1/3}. \tag{3.10}$$

For arbitrary $\kappa=k/n$, Eq. (3.3) is not difficult to solve numerically. In this case it turns out that $F_*(\kappa)$ and $\epsilon_*(\kappa)$ increase monotonically as κ increases; they depend strongly on κ near $\kappa=1$, and are almost constant for $\kappa<0$, which qualitatively agrees with the expansions (3.8) and (3.9). In fact, the coefficients c_{11} and c_{12} (for the terms that are singular at the point $\kappa=1$) are an order of magnitude larger than c_9 and c_{10}. Recall that $\kappa=1-n^{-1}$ corresponds to the states $(n-1,0,0)$, while $\kappa=-1+n^{-1}$ goes with the states $(0,n-1,0)$.

It follows from Eq. (3.1) that the reduced energy is singular for $F=F_*$, and develops an imaginary part when continued into the region $F>F_*$:[4)]

$$\epsilon=\epsilon_*(\kappa)\left\{1+af+b\frac{f}{\ln(-f)}+\ldots\right\},$$

$$\epsilon''=-\operatorname{Im}\epsilon=\text{const}\cdot f(\ln f)^{-2}\theta(f), \tag{3.11}$$

where $f=(F-F_*)/F_*\to 0$, $\theta(x)=(1/2)(1+\operatorname{sgn}x)$ and the coefficients a,b depend on κ. Thus, for $\kappa=-1$ we have $a=1/2$, $b=2/3$; on the other hand, for $\kappa=1$ the singularity in (3.11) vanishes (see the next section).

In conclusion, we note that for $m\neq 0$ it is not possible to obtain analytic expressions for the classical ionization threshold[5)] similar to Eqs. (3.3) and (3.7). The problem of computing $F_*(n_1,n_2,m)$ has been discussed by a number of authors.[30,9,28]

4. A SPECIAL CASE: THE RYDBERG STATES $(n-1,0,0)$

Let us pause to discuss this special case, since these states (and states that are close to them with regard to quantum number, e.g., $(n-2,1,0)$ with $n\gg 1$) are the longest-lived of the states with a given n, and are observed during photoionization of atoms in a constant electric field.[1-3]

We will show that in this case, the solution to Eq. (3.1) has no singularities when $F=F_*$. We note above all that the system (3.1) has the solution $\beta_1(F)=1$, $\beta_2(F)=0$ for $\nu_1=1$, $\nu_2=0$. Because of this, the system reduces to the first equation (since the second is satisfied identically). By using relations for the hypergeometric function, this equation can be written in the form

$$\frac{1}{2}(\cos t)^{1/2}\,{}_2F_1\left(1/2,5/2;2;\frac{1+\sin t}{2}\right)=F^{1/4},$$

$$\epsilon=4F^{1/2}\tan t,\quad -\frac{\pi}{2}\leqslant t<\frac{\pi}{2}, \tag{4.1}$$

which specifies the function $\epsilon(F)$ in parametric form. From (4.1) it is obvious that the energy has no singularities and remains real for all values of F, $0<F<\infty$:

$$\epsilon(F)=\begin{cases} -1+3F-\frac{7}{4}F^2+\frac{33}{8}F^3-\frac{465}{32}F^4+\ldots, & F\to 0,\\ c_{13}(f-c_{14}f^2+c_{15}f^3-c_{16}f^4+\ldots), & F\to F_*,\\ (3\pi F)^{2/3}[1-c_{17}F^{1/3}(\ln F+c_{18})+\ldots, & F\to\infty. \end{cases} \tag{4.2}$$

Here[31]

$$c_{13}=\frac{64}{27\pi}K^2,\quad c_{14}=\left(\frac{1}{8}-\frac{1}{6}K^2\right),$$

$$c_{15}=\left(-\frac{3}{32}+\frac{1}{36}K^2+\frac{1}{18}K^4\right) \tag{4.3}$$

etc., while

$$F_*(1)=\lim_{n\to\infty} F_c(n-1,0,0)$$

corresponds to $t=0$. At this point, let us make a few remarks.

1) The perturbation expansion for $\epsilon(F)$ converges for $0<F<F_*$.

2) The expansion coefficients decrease rapidly for $F\approx F_*$ (see Table III); therefore, the dependence of the resonance energy E_r on electric field in the vicinity of the ionization limit $E_r=0$ is always close to linear. This is easy to see from the figures given in Refs. 28, 32, which serve to confirm the scaling relationships both for below-barrier and above-barrier resonances.

3) The asymptotic dependence $\epsilon(F)\propto F^{2/3}$ for ultrastrong fields is associated with the form of the external potential $\mathscr{E}z$ acting on an atomic electron. This asymptotic form is easy to explain using the Bohr–Sommerfeld quantization rules analytically continued to the above-barrier region[29] (see also Ref. 33 for the case of a short-range potential and a low-frequency field).

The coefficients c_{17}, c_{18} in (4.2) determine the corrections to the leading term $(3\pi F)^{2/3}$ of the asymptotic series. The leading correction contains $\ln F$ along with $F^{-1/3}$. However, since $c_{18}\approx 16\gg 1$, for practically achievable fields the correction follows a power law ($\propto F^{-1/3}$).

4) According to (4.2), the function $\epsilon(F)$ never develops an imaginary part; on the other hand, for finite n we have $\Gamma_n>0$. However, there is no contradiction here, since the widths Γ_n for the states $n-1,0,0$ are $\propto 1/n$, i.e., of higher order of smallness, implying that ϵ_n'' vanishes in the limit $n\to\infty$.

5) Of all the states with given n, the state $(0,n-1,0)$ has the lowest classical ionization threshold, while the state $(n-1,0,0)$ has the highest.

5. STARK RESONANCES FOR $\mathscr{E}=\mathscr{E}_c$

The quantities of interest when a level "touches" the top of the potential barrier (i.e., for $\mathscr{E}=\mathscr{E}_c$) are the position $E_r^{(n_1n_2m)}$, and even more so the width, of the Stark resonances. In order to solve this problem it is natural to use modified Bohr–Sommerfeld quantization rules that take into account the permeability of the barrier.[10,11] In the Stark effect problem, the potential $U_1(\xi)$ is closed off, while $U_2(\eta)$ has a barrier.[13] Accordingly, the quantization rules take the form

$$J_1=\nu_1,\quad J_2=\nu_2-\frac{1}{2\pi n}\varphi(a), \tag{5.1}$$

where

$$J_i=\frac{1}{\pi}\int_{q_0}^{q_1}k_i(q),$$

$$k_i=\left\{\frac{\epsilon}{4}+\frac{\beta_i}{q}-\frac{\mu^2}{4q^2}+\frac{(-1)^i}{4}Fq\right\}^{1/2}. \tag{5.2}$$

Here $q=x$ or y (for $i=1,2$ respectively); q_j are turning points ($j=0,1,2$), and $q_1<q<q_2$ is the sub-barrier region, for which $k_2^2<0$. Finally, we have $x=n^{-2}\xi$, $y=n^{-2}\eta$, where ξ,η are parabolic coordinates. The function $\varphi(a)$ in (5.1) is related to the finite permeability of the barrier in the potential $U_2(\eta)$, and equals

$$\varphi(a)=\frac{1}{2i}\ln\left\{\Gamma\left(\frac{1}{2}+ia\right)\Big/\Gamma\left(\frac{1}{2}-ia\right)(1+e^{-2\pi a})\right\}$$
$$+a(1-\ln a). \tag{5.3}$$

It depends on the parameter a, a quantity that will be extremely useful in what follows:

$$a=\frac{n}{\pi}\int_{y_1}^{y_2}[-k_2^2(y)]^{1/2}dy. \tag{5.4}$$

For states with energies $E_r\approx U_m$ we can expand the quantization integrals J_i near the top of the barrier, i.e., letting $F\to F_c$, $y\to y_m$, and $a\to 0$. In the linear approximation ($\epsilon''=-\mathrm{Im}\,\epsilon\to 0$) we obtain

$$\beta''=-\mathrm{Im}\,\beta_2=\frac{s}{4}\epsilon'',\quad a=i\frac{n}{8\omega}\left(1+\frac{s}{y_m}\right)\epsilon'', \tag{5.5}$$

where

$$s=\frac{\sigma_0}{\sigma_1},\quad \sigma_j=\int_{x_0}^{x_1}\frac{dx}{x^jk_1(x)},\quad j=0,1. \tag{5.6}$$

Because the partial derivatives $\partial J_2/\partial\epsilon$ and $\partial J_2/\partial\beta_2$ diverge logarithmically as $F\to F_c$, it is more tedious to calculate the expansion of the second of Eqs. (5.1). Here we omit these calculations and simply give the final expressions.[6] We will use a bar on top to denote the level width at the point where the level touches the top of the barrier [i.e., $\bar\Gamma\equiv\Gamma_n$ ($\mathscr{E}=\mathscr{E}_c$)], and use this notation in discussing other quantities evaluated at $\mathscr{E}=\mathscr{E}_c$. Then

$$\bar\Gamma_{n_1n_2m}=\frac{\gamma}{n^3}[(\ln(n_2+1/2)+l_0]^{-1},\quad n\gg 1, \tag{5.7}$$

where γ and l_0 are constants that depend on ν_1, ν_2, and μ (but do not depend on n):

$$\gamma=\frac{4\omega\ln 2}{1+s/y_m}, \tag{5.8}$$

$$l_0=C+\ln[8\omega(y_m-\bar y_0)^2]+\frac{2(\tau_0+s\tau_1)}{1+s/y_m}, \tag{5.9}$$

$$\tau_j=\int_{\bar y_0}^{y_m}dy\left\{\frac{\omega}{y^j\bar k_2(y)}-\frac{1}{y_m^j(y_m-y)}\right\},\quad j=0,1, \tag{5.10}$$

$C=0.5772...$, while ω is the frequency of the "inverted oscillator," i.e.,

$$\tilde{k}_2(y)=\omega(y_m-y)+..., \quad y\to y_m. \tag{5.11}$$

Taking (5.11) into account, we see that the second term in the expression under the integral sign in (5.10) regularizes the integral, which diverges at the upper limit $y=y_m$. This regularization makes the integrals τ_j converge, so that they can be evaluated numerically. For $m=0$ states, the integrals σ_j and τ_j can be computed analytically:

$$s(\kappa)=(3\pi)^2\frac{(1-\kappa)^2}{256}z(1+z)^{1/2}\frac{F[3/4,3/4;2;z/(1+z)]}{F[1/4,1/4;1;z/(1+z)]}, \tag{5.12}$$

$$\tau_0=-2(1-\ln 2), \quad \tau_1=2\ln 2/y_m, \tag{5.13}$$

(see Appendix C), where $z=z(\kappa)$ is defined in Sec. 3, while the expression for the maximum point y_m is given in (3.7).

All that remains is for us to determine the frequency ω. For this we note that as $u\to u_m$ we have $p^2(u)=\bar{\omega}^2(u-u_m)^2$ in (2.3); here $p=k_2(y)/y$, where p and k_2 are the semiclassical momenta in (2.3) and (5.2). From this we have

$$\omega=\frac{\bar{\omega}}{y}=\left[\frac{1}{2y^3}\frac{d}{dy}\left(y\frac{d(p^2)}{dy}\right)\right]^{1/2}_{y=y_m}$$

$$=\frac{3(1-\zeta)^{1/4}}{2[1+(1-\zeta)^{1/2}]}(-\epsilon_c')^{-1/2}F_c \tag{5.14}$$

(ω refers to the reduced variable y, and therefore differs from the frequency of the "inverted oscillator" for the potential $U_2(\eta)$ by a factor of n^3; this factor is given in (5.7) in explicit form).

Since Eq. (5.7) and those that follow it are asymptotic, the parameters entering into them should be computed for $n\to\infty$. This implies that in (5.14) we must substitute $F_c\to F_*$, $\epsilon_c'\to\epsilon_*$. For states with $m=0$ we obtain

$$\omega=\frac{2^9}{(3\pi)^3}(1-\kappa)^{-3}(1+z)^{-2}. \tag{5.15}$$

Equations (5.11)–(5.15) give all the quantities entering into the asymptotic form (5.7) for $\bar{\Gamma}_n$. In particular, for the states $(n_1,n_2,0)$ we have $\kappa\to-1$, $s=3\pi\cdot 2^{-7/2}(1+\kappa)+...$ when $n_1\ll n_2$, and, taking (3.6) into account, we find

$$\gamma=\frac{2^8\ln 2}{(3\pi)^3}[1+c_{19}(1+\kappa)+...],$$

$$l_0=C+\ln 96\pi-4+c_{20}(1+\kappa)+..., \tag{5.16}$$

where

$$c_{19}=\frac{3}{2}\left(1-\frac{2^{3/2}}{\pi}\right), \quad c_{20}=\frac{1}{2}(c_4-1), \tag{5.17}$$

with $1+\kappa=(2n_1+1)/n$. As n_1 increases, the constant γ in (5.7) increases, which agrees with Fig. 3 and with the numerical calculations.

It should be noted that the asymptotic form (5.7) can be modified if $n_2=0$. Thus, let us consider the "circular" states $(0,0,n-1)$. Both numerical and analytic calculations show that in this case

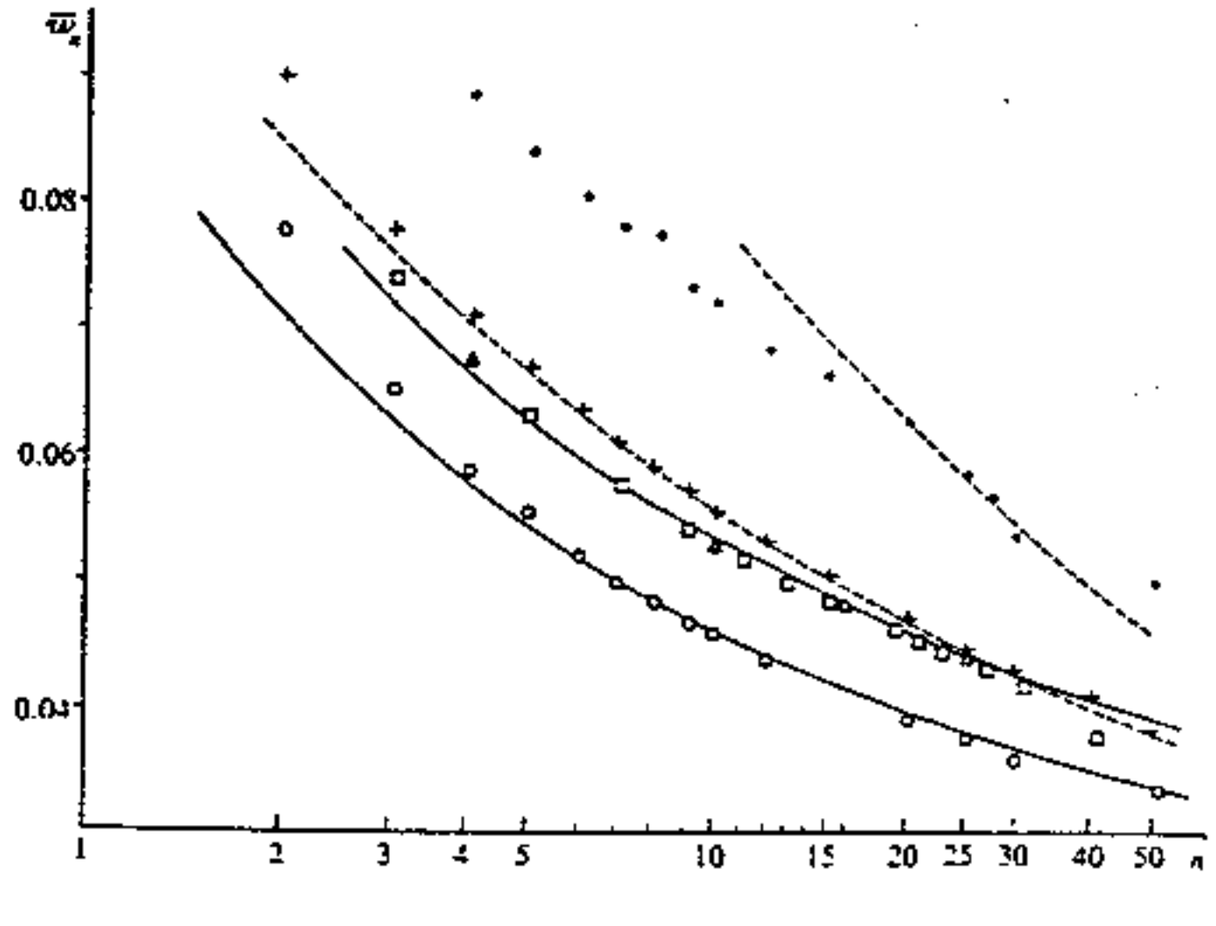

FIG. 3. The quantities $\bar{w}_n=n^3\bar{\Gamma}_n$ (for $\mathscr{E}=\mathscr{E}_c$) for a hydrogen atom. Results of numerical calculations: ○ denotes states $(0,n-1,0)$, + denotes $(0,0,n-1)$, ● denotes $(n-1,0,0)$, □ denotes $(n_1,n_1,0)$, and △ denotes the series (m,m,m), where $n_1=(n-1)/2$ and $m=(n-1)/3$. The solid curves were plotted using Eq. (5.7) with values of the parameter γ and l_0 given in Table II for $\kappa=-1$ and 0; the dashed curves correspond to the asymptotic forms (5.18) and (5.19).

$$\bar{\Gamma}^{(0,0,n-1)}\approx\gamma_1 n^{-16/5}, \quad n\gg 1. \tag{5.18}$$

For the states $(n-1,0,0)$ we find that $\omega\propto(1-\kappa)^{-1/3}$, $y_m\propto(1-\kappa)^{2/3}$, while s goes to a constant value. Substituting this into (5.8), we find that $\gamma\propto(1-\kappa)^{1/3}\propto n^{-1/3}$, so that $1-\kappa=(2n_2+1)/n$. Thus,

$$\bar{\Gamma}^{n-1,0,0}\approx\gamma_2 n^{-10/3} \tag{5.19}$$

[in this case the constants γ_1 and γ_2 differ from γ in Eq. (5.7)]. In the case where (5.19) holds, the Stark widths $\bar{\Gamma}_n$ decrease most rapidly. This indicates that the states $(n-1,0,0)$ are the longest-lived of all the states with fixed n. For a derivation of Eqs. (5.18) and (5.19), see the next section.

Let us compare the asymptotic expressions we have obtained with numerical calculations. The widths $\bar{\Gamma}_n$ (for $\mathscr{E}=\mathscr{E}_c$) were calculated for various states (n_1,n_2,m) of a hydrogen atom by summing the perturbation expansion (by the Padé–Hermite approximant method, see Ref. 28). As is clear from Fig. 3, the asymptotic expression (3.7) is in good agreement with the results of numerical calculations, even for $n>3$ (the scatter of points for $n>30$ is due to poor convergence of the Padé–Hermite approximant method, which can only be improved by introducing yet higher orders of perturbation theory into the calculation). The agreement between (3.7) and the numerical calculations is even surprising if we note that the asymptotic form (3.7) was itself obtained in the logarithmic approximation, i.e., under the condition $\ln(n+1/2)\gg 1$. This is most likely a manifestation of the fact that the region of applicability of the semiclassical expressions in physical problems usually "stretches" down to values $n\sim 1$ (in this connection see Ref. 27). The power-law asymptotic forms (5.18) and (5.19) also agree qualitatively with numerical calculations, although their accuracy is some-

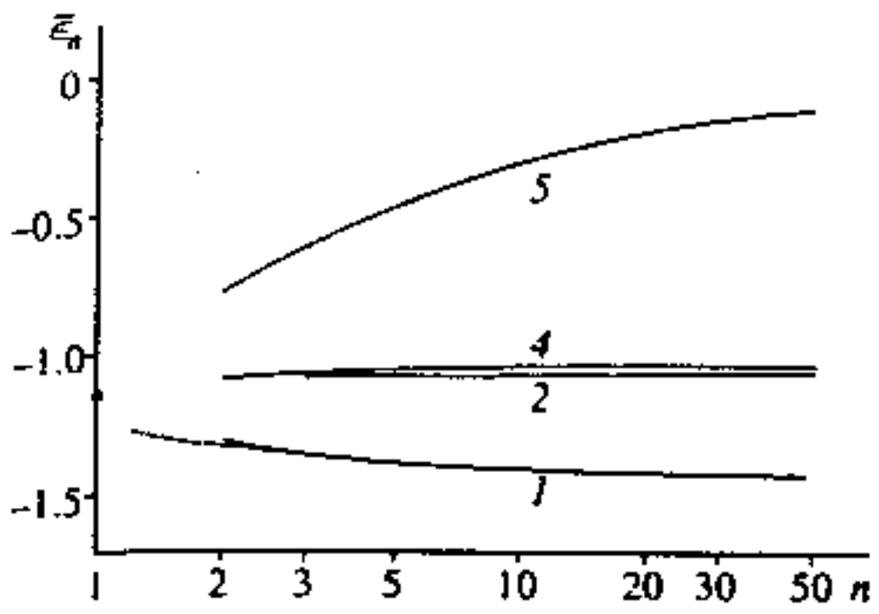

FIG. 4. Reduced energies $\bar{\epsilon}_n$ for $\mathscr{E}=\mathscr{E}_c$. The notation for the curves is the same as in Fig. 1. The dashed curve corresponds to the asymptotic form (A7) for the states $(0,n-1,0)$; in this case $\bar{\epsilon}_n = \epsilon'_n$ in (A7).

what lower. This is also understandable, for in (5.19), for example, the expansion parameter is $n^{-2/3}$, and not $1/n$.

The widths $\bar{\Gamma}_n$ themselves are shown in Fig. 2 of Ref. 23, from which it is clear that the Stark levels for $\mathscr{E}=\mathscr{E}_c$ have even smaller widths, especially in the case of Rydberg states.

Let us now consider the real part of the resonance energy $\bar{E}_r$ when $\mathscr{E}=\mathscr{E}_c$. Figure 4 shows the results of numerical calculations of the reduced energy $\epsilon_n = 2n^2E_r^{(n_1n_2m)}$ for several series of values (n_1,n_2,m), obtained by the same method as for the case of $\bar{\Gamma}_n$. As a rule, ϵ_n depends weakly on n, and for $n\to\infty$ it passes rather rapidly to a limiting value equal to ϵ_* (as usual, the exceptions are the states $[n-1,0,0)]$.

In addition to F_c, another characteristic field is F_0, the field at which the quasistationary level intersects the boundary $E_r=0$, i.e., the ionization limit for the free atom (in the absence of an external field, $\mathscr{E}=0$). In Figs. 2 and 5 we show the values of F_0 and the corresponding widths $\bar{\Gamma}_n(F=F_0)$ for the states $(n-1,0,0)$ and states that are close to them with regard to quantum numbers.[7)] Although the fields $F_0(n-1,0,0)$, like $F_c(n-1,0,0)$, all have the common limit $F_*(1)\approx 0.383$ as $n\to\infty$, for $n\sim 30$ they are still rather

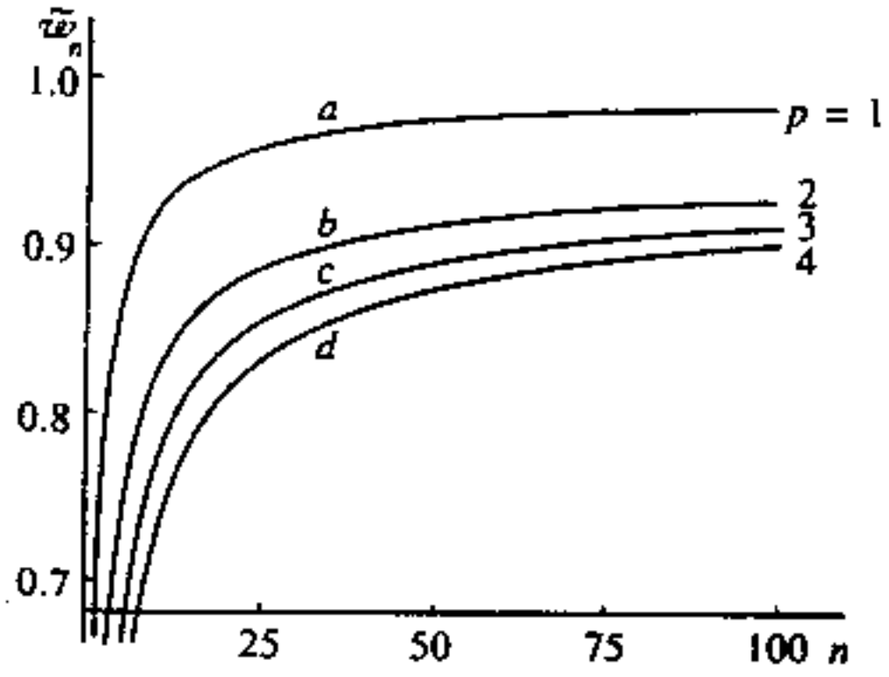

FIG. 5. The quantities $\tilde{w}_n = (\bar{n}/p)^{4/3}\epsilon''_n$ for $F=F_0$, i.e., where the levels intersect the boundary of the continuum [compare with Eq. (5.20)]. Notation of the curves is the same as for Fig. 2, $\bar{n}=n-p/2$; for these curves the values of p are indicated.

far from this value, as we see from Fig. 2. As the electric field increases from F_c to F_0, the widths Γ_n increase by one to two orders of magnitude; nevertheless, the Rydberg states remain rather narrow for $F=F_0$. This explains the fact that in the range $E_r>0$, five to ten resonance peaks are usually observed[2,3] (and sometimes several dozen peaks[1,7]) before they overlap.

Numerical fitting shows that for $n_1\gg n_2$, $m\sim 1$, and $F=F_0$, the following relation holds (see Fig. 5):

$$\bar{\Gamma}^{(n_1n_2m)}\approx \text{const}\cdot n^{-2}\left(\frac{p}{n-p/2}\right)^{4/3}, \quad p=2n_2+|m|+1, \tag{5.20}$$

which generalizes Eq. (5.19) and the scaling relations for above-barrier resonances (see Eq. (28) in Ref. 32). From this it is clear that the widths $\bar{\Gamma}_n$ rapidly increase as the quantum numbers n_2 and m increase, which distinguishes this case from the levels $\bar{\Gamma}_n$ when $\mathscr{E}=\mathscr{E}_c$.

6. ASYMPTOTIC BEHAVIOR OF STARK WIDTHS (DEGENERATE CASE)

For states with $n_2=0$, the semiclassical approximation is no longer applicable, for which reason the asymptotic expression (5.7) is modified. Let us briefly discuss this case.

For the circular states $(0,0,n-1)$ the effective potential $U_2(\eta)$ has an inflection point for $F=F_*$ and $n\to\infty$; therefore, when $F\approx F_*$, $\epsilon\approx\epsilon_*$, the Schrödinger equation takes the form

$$\left(-\frac{1}{2n^2}\frac{d}{dx^2}+x^3+fx-e\right)\chi_2=0, \tag{6.1}$$

where $x\propto\eta-\eta_*$, $e\propto\varepsilon-\varepsilon_*$ and $f\propto F-F_*\to 0$ (here we omit some unimportant numerical factors). We can also study the more general case where the x^3 in (6.1) is replaced by x^N with odd $N=3,5,\ldots$ (an inflection point when $f=0$). After rescaling,

$$x=n^{-2/N+2}y, \quad f=n^{-2(N-1)/N+2}g, \quad e=n^{-2N/(N+2)}E, \tag{6.2}$$

Eq. (6.1) reduces to the canonical equation

$$\left(-\frac{1}{2}\frac{d^2}{dy^2}+y^N+gy-E\right)\chi=0, \tag{6.3}$$

which determines the dependence of the reduced energy E on g (and does not contain n explicitly). There is a potential barrier for values of the effective coupling constant $g<0$, which vanishes when $g\geqslant 0$. Therefore, there exists a certain $g=g_c<0$ for which $\operatorname{Re}E(g_c)$ coincides with the maximum of the potential y^N+gy (the point at which the level "touches" the top of the barrier; the numerical solution for $N=3$ gives $g_c=-1.013$, $E(g_c)=0.392-i0.096$). In the original Schrödinger equation, this corresponds to the asymptotic forms

$$\epsilon-\epsilon_*\propto n^{-2N/(N+2)}, \quad F_c-F_*\propto n^{-2(N-1)/(N+2)}, \quad n\to\infty, \tag{6.4}$$

from which Eq. (5.18) follows for $N=3$.

Let us turn to the states $(n-1,0,0)$. From (5.15) and (3.6) it follows that the frequency $\omega \propto (1-\kappa)^{-1/3}$, and $y_m \propto (1-\kappa)^{2/3}$ for $1-\kappa = 1/n \to 0$. In (5.12) we have $z/(z+1) \to 1$, so that the parameter s goes to the constant value:

$$s(1) = \pi \left(\frac{3}{4K} \right)^2 \equiv c_{21}. \tag{6.5}$$

Taking this into account, (5.8) implies that $\gamma \propto (1-\kappa)^{1/3}$, and we arrive at (5.19). Thus, the fact that the asymptotic forms (5.18) and (5.19) differ from the general case (5.7) is associated with the inapplicability of the semiclassical method with regard to the variable η (along which real tunneling takes place). Note, however, that the exponents 16/5 and 10/3 in these asymptotic forms differ negligibly from the exponent 3 in (5.7).

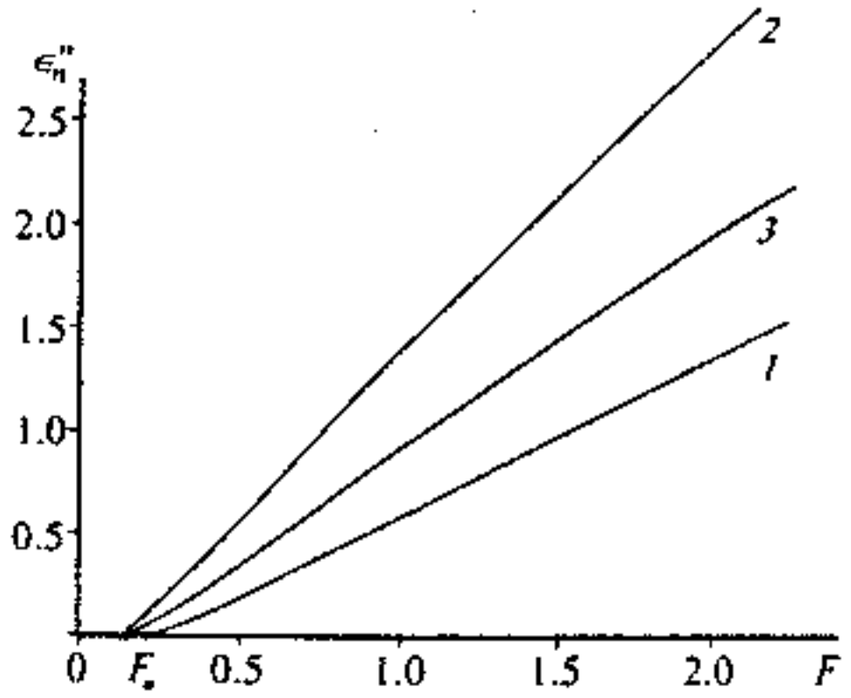

FIG. 6. The degree of linearity of the reduced energy ϵ_n'' in the above-barrier region, $F > F_*$. Curves *1* and *2* are for the Rydberg states $(0,0,n-1)$ and $(0,n-1,0)$, curve *3* for the spherical model (6.5). The values of F_* are respectively: (*1*) 0.208, (*2*) 0.130, and (*3*) 0.145.

7. ABOVE-BARRIER STARK RESONANCES

At this time various numerical methods have been developed that can be used to calculate the positions E_r and widths Γ of hydrogenic Stark resonances to high accuracy. The calculations show[9,15] that there is a range of fields $\mathscr{E} > \mathscr{E}_c$ in which the widths of the resonances are essentially linear in the field strength:

$$\epsilon''(F) \equiv n^2 \Gamma_n = c(F - \tilde{F}_0), \quad F > 1.2 F_c \tag{7.1}$$

(the constants c and $\tilde{F}_0$ depend on quantum numbers). In order to explain this fact we use the semiclassical $1/n$ expansion:[16,19]

$$\epsilon_n(F) = \epsilon^{(0)} + \frac{\epsilon^{(1)}}{n} + \ldots + \frac{\epsilon^{(k)}}{n^k} + \ldots, \tag{7.2}$$

whose leading term $\epsilon^{(0)}$ corresponds to the energy of an electron in the equilibrium classical orbit. The higher orders in the $1/n$ expansion $\epsilon^{(k)}$ can be evaluated recursively (in prac-

(here $\rho = \sin 8\tau / \sin 9\tau$, $0 \leqslant \tau < \pi/9$; the value $\tau = 0$ corresponds to the classical ionization threshold F_*, $\tau \to \pi/9$ to the case $F \to \infty$).

We can discuss the "spherical model" for the Stark effect in hydrogen analogously:

$$V(r) = -\frac{1}{r} - \mathscr{E} r, \quad 0 < r < \infty. \tag{7.5}$$

For states with angular momentum $l = n-1 \ll 1$, we can use the $1/n$ expansion to derive equations of the same type, but simpler:

$$\epsilon'' = 4 \sin^4 \tau \sin 2\tau (\sin 3\tau)^{-2},$$
$$F = \sin \tau \sin^2 2\tau (\sin 3\tau)^{-3}, \tag{7.6}$$

where now $F_* = 4/27$, and $0 \leqslant \tau < \pi/3$. The calculations using these expressions are elementary, and were used to plot the curves *1* and *3* in Fig. 6; for $F > 1.2 F_*$ these curves are extremely close to straight lines ($\sigma < 0.005$, see Table IV)

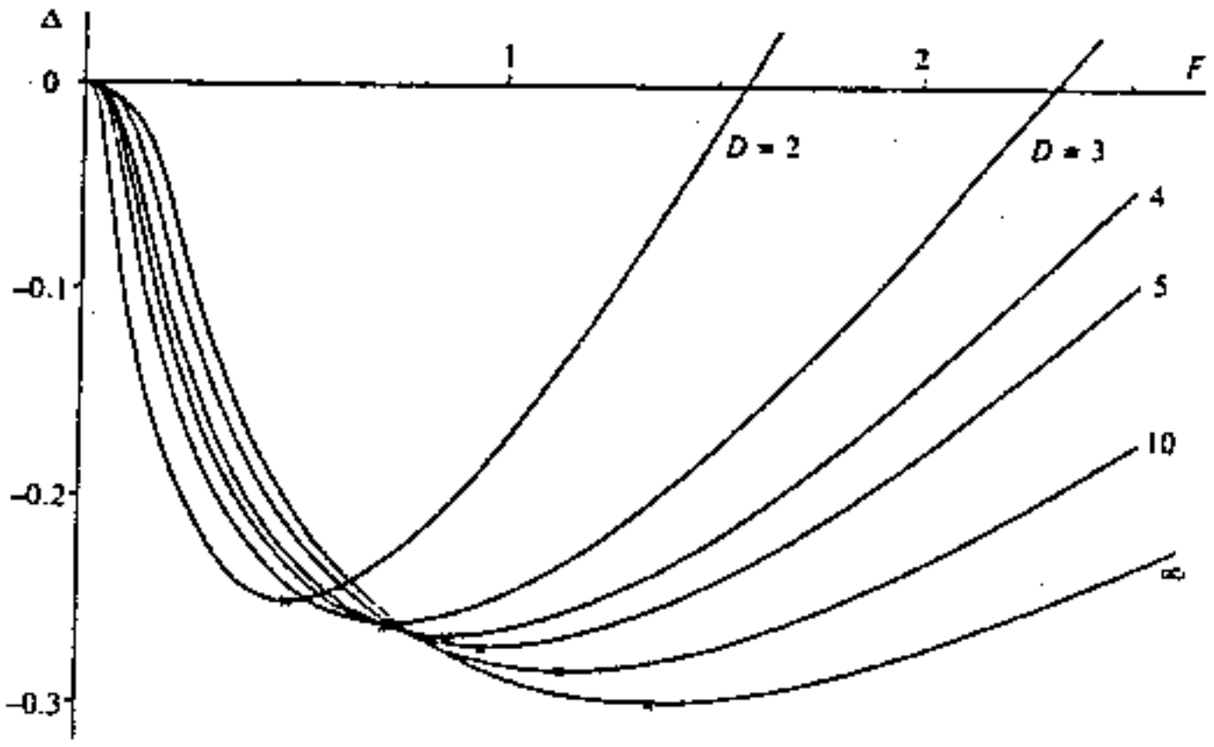

FIG. 8. Stark shift Δ of the ground state of a "hydrogen atom" for various spatial dimensions D. The position of the maximum shift is indicated by the cross.

$$\Delta_m = 1 - 3^{3/2} \cdot 2^{-2} \approx -0.299. \tag{8.6}$$

It follows from (7.3) that in the limit $D \to \infty$ we have

$$\frac{d\epsilon^{(0)}}{dF} = -2Fu^{-6}, \quad \frac{d^2\epsilon^{(0)}}{dF^2} = \frac{1 - \frac{3}{4}u}{u^6\left(1 - \frac{9}{8}u\right)}. \tag{8.7}$$

When $F < F_*$, setting $\epsilon^{(0)} = \epsilon' - i\epsilon''$, we find

$$\frac{d\epsilon'}{dF} = -2F\left(\frac{\sin 9\tau}{\sin 8\tau}\right)^6 \cos 6\tau. \tag{8.8}$$

Thus, the reduced energy $\epsilon'(F)$ has a minimum for $\tau = \pi/12$ (or $F = F_m$), while the Stark shift reaches its maximum value at this point. At this point all the parameters are calculable analytically: $u = u_m = (3/2)^{1/2} \exp(-i\pi/12)$, which gives (8.6) and a slope for $\epsilon''(F)$ equal to

$$c_m = \left.\frac{d\epsilon''}{dF}\right|_{F=F_m} = \frac{2^{3/2}}{3}(\sqrt{3}-1)^{1/2} = 0.807, \tag{8.9}$$

where $F_0/F_m = (\sqrt{3}-1)/4$; F_m is the point where $\epsilon'(F)$ is a minimum, while $\tilde{F}_0$ is a parameter obtained from the interpolation formula (7.1). We note that the coefficient c_m is quite close to its optimal value from numerical fitting (see Table IV). The corresponding curves for the reduced width of the ground state $\epsilon'' = ([D-1]/2)^2\Gamma(F,D)$ confirm the linear dependence (6.1) in the region $F > 1.2F_*$.

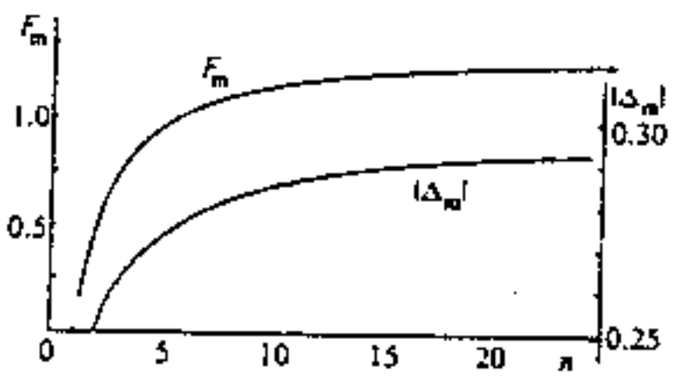

FIG. 9. Maximum shift of the ground state level and the corresponding field $F_m(D)$. Reduced variables (8.5) are used here, $n = (D-1)/2$.

Equations (8.6)–(8.9) apply to the limiting case $D = \infty$. Figure 9 shows the dependence of F_m and Δ_m on the dimension of the space D obtained using the numerical methods described above.

In conclusion, we note that for odd dimensions, the results given above are identical to those for the circular states $(0,0,n-1)$ of a hydrogen atom in ordinary (three-dimensional) space, where $n = (D-1)/2$; however, for $D = 2,4,6,\ldots$ they correspond to half-integer values of the principal quantum number n, which cannot occur in the three-dimensional problem.

9. CONCLUSION

Of the various results of this work, the most fundamental are probably Eqs. (5.7), (5.18), and (5.19), which predict that the Stark resonances for $\mathscr{E} = \mathscr{E}_c$ are still narrow, especially for $n \gg 1$ (see also Fig. 2 in Ref. 23).

At first glance, this appears to contradict the well-known result of Kemble: namely, that the coefficient for penetration through a parabolic barrier at the point where the level touches the top of the barrier ($E_r = U_m$) is $\bar{D} = 1/2$, independent of frequency. The same is also true of other smooth potentials, for example, for

$$U(x) = U_m/\mathrm{ch}^2(x/L), \quad -\infty < x < \infty. \tag{9.1}$$

In this case[11]

$$\bar{D}(E = U_m) = \left\{1 + \frac{\mathrm{ch}^2\pi(Q^2 - 1/4)^{1/2}}{\mathrm{sh}^2\pi Q}\right\}^{-1}, \tag{9.2}$$

where $Q = (2U_mL^2)^{1/2} = 2U_m/\omega$, from which we find

$$\bar{D} = \begin{cases} 1 - Q^2 - \frac{1}{2}(\pi^2 - 4)Q^4 + \ldots, & Q \to 0 \\ \frac{1}{2} + \frac{\pi}{16Q} + O(Q^{-3}), & Q \gg 1. \end{cases} \tag{9.3}$$

The parabolic approximation is valid for $|x| \ll L$, while the condition for applicability of the semiclassical method is

$$\left|\frac{d}{dx}\left(\frac{1}{\bar{p}}\right)\right| = \left(Q\ \mathrm{sh}^2\frac{x}{L}\right)^{-1} \ll 1.$$

Both conditions are fulfilled for $LQ^{-1/2} \ll x \ll L$, which requires $Q \gg 1$. However, in this case the coefficient of penetration approaches the Kemble value $\bar{D} = 1/2$ and is not small at all.

Let us clarify the reason for the smallness of the widths $\bar{\Gamma}_n$ for $n \gg 1$. It is obvious that in the parabolic approximation, $\bar{\Gamma}_n \sim \omega$, simply because there is no other parameter with the same dimensionality in the problem.

We now show that the frequency ω of the "inverted oscillator" itself depends on the quantum number n. Specifically,

$$\omega \propto n^{-(2+\alpha)/(2-\alpha)}, \tag{9.4}$$

if the potential that binds the particle $V(r) \propto r^{-\alpha}$ at small distances. This is not difficult to verify for the following example:

$$V(r)=-\frac{g}{\alpha r^{\alpha}}-\mathscr{E}r, \quad 0<\alpha<2 \tag{9.5}$$

($l=0$). If $\mathscr{E}\to 0$, the levels with energies $E\sim V_m\propto \mathscr{E}^{\alpha/(\alpha+1)}$ are located in the region of highly excited states that are close to the continuum (the point where the potential is a maximum $r_m=(g/\mathscr{E})^{1/(\alpha+1)}\to\infty$). In the semiclassical approximation

$$|E_n|\sim n^{-2\alpha/(2-\alpha)}, \quad \langle r_n\rangle\sim n^{2/(2-\alpha)}, \quad n\gg 1. \tag{9.6}$$

Taking into account that $\omega\sim\langle T_n\rangle^{1/2}/\langle r_n\rangle$, in which $\langle T_n\rangle=[\alpha/(\alpha-2)]E_n$ (the virial theorem), we arrive at the estimate (9.4). If $\alpha=1$ (i.e., a Coulomb singularity for $V(r)$ as $r\to 0$), then $\omega\propto n^{-3}$, which coincides with the collision frequency of a particle localized in a well, and (5.7) corresponds to collision with the barrier wall.

We note that (5.7) contains still another order of smallness $\sim 1/\log n$ for the widths $\tilde{\Gamma}_n$. The origin of this effect is as follows. According to Gamow's formula, $\Gamma_n\propto T^{-1}\exp(-2\pi a)$, where T is the period of oscillation of a classical particle in the well. As the energy approaches the top of the barrier, $a=(U_m-E)/\omega\to 0$ and the widths Γ_n are no longer exponentially small; however, the period T diverges logarithmically:

$$T\sim\omega^{-1}\ln\frac{1}{a}, \quad \Gamma_n\sim T^{-1}\propto\frac{1}{n^3\ln n} \tag{9.7}$$

[the parameter $a\propto n$, which is apparent from (5.4) and (5.5)]. Although the factor $(\log n)^{-1}$ is not very important compared to n^{-3}, it is fully confirmed by numerical calculation (Fig. 3). In addition, it determines the parametric smallness of the ratio $\Gamma_n/\Delta E_n\propto(\ln n)^{-1}$, which causes the Stark resonances to remain isolated for $F=F_c$ (especially in the case $n\gg 1$).

The authors are grateful to N. B. Delon, B. M. Karnakov, and V. P. Krainov for useful discussions, and to S. G. Pozdnyakov and D. V. Popov for assistance in the numerical calculations. The work was carried out with partial support from the Russian Fund for Fundamental Research (Project 93-02-14368) and the International Science Fund (ISF grant Ph 1-2292-0925).

APPENDIX A

Here we derive the asymptotic form (2.11) for the states $(0,n-1,0)$ with $n\gg 1$ ($\nu_1=1/2n\to 0$, $\nu_2=1-1/2n$).

In the semiclassical approximation the quantities $\epsilon=\epsilon'-i\epsilon''$, β_1 and β_2 are found from the quantization rules taking into account the permeability of the barrier, see Eq. (5.1). The function $\varphi(a)$, defined by Eq. (5.3), has the following expansion as $a\to 0$

$$\varphi(a)=i\frac{\ln 2}{2}-a\ln a+\sum_{k=1}^{\infty}d_k a^k, \tag{A1}$$

$$d_1=1-2\ln 2-C-i\frac{\pi}{2}, \quad d_2=i\frac{\pi^2}{4},\ldots, \tag{A2}$$

where $C=0.5772...$ is the Euler constant. In this case

$$a=\frac{1}{\pi}\int_{\eta_1}^{\eta_2}(-p_\eta^2)^{1/2}d\eta=n\frac{(-\epsilon)^{3/2}}{2^{7/2}F}G(1-z_2), \tag{A3}$$

where

$$G(z)=zF(1/4,3/4;2;z)$$

$$=\begin{cases} z+\frac{3}{32}z^2+\ldots, & z\to 0 \\ c_4[1+\frac{3}{16}t(\ln t-c_{22})+\ldots], & z=1-t\to 1, \end{cases} \tag{A4}$$

and $c_{22}=1+6\ln 2$. It follows from (2.8) that at the point where the level touches the top of the barrier,

$$16\beta_2'F/(-\epsilon')^2=1, \tag{A5}$$

or $\zeta=3/4$. Since $\beta_1\propto 1/n$, in the first equation from the system (3.1) the variable $z_1\propto 1/n\to 0$, so that

$$\beta_1=\frac{1}{2n}\left[(-\epsilon')^{1/2}-i\frac{\epsilon''}{2\epsilon'}+O\left(\frac{1}{n^2}\right)\right]$$

(from what follows it is clear that $\epsilon''\ll|\epsilon'|$). Substituting this expansion into $z_2=16(1-\beta_1)F/\epsilon^2$ and taking (A5) into account, we obtain

$$z_2=1-2i\epsilon''/(-\epsilon')+\ldots\to 1. \tag{A6}$$

Using (A4) in the case $z\to 1$, we reduce the remaining equation to the following form:

$$\frac{(-\epsilon)^{3/2}}{3\pi\sqrt{2}F_c}\left\{1-\frac{3}{16}\left[\ln(96\pi n)+C+\frac{i\pi}{2}\right](1-z_2)+\ldots\right\}$$

$$=1-\frac{1}{2n}-i\frac{\ln 2}{4\pi n}+\ldots$$

(it should be noted that the logarithmic terms in (A1) and (A4) cancel without a remainder). Substituting (A6) at this point and separating its real and imaginary parts, we finally obtain

$$\epsilon_n'=\epsilon_*(-1)\left[1-\frac{c_{10}}{n}+\frac{c_2}{2n(\ln n+l_0)}+\ldots\right],$$

$$\epsilon_n''=\gamma/n(\ln n+l_0), \quad \epsilon_*(-1)=-2^7\cdot(3\pi)^{-2} \tag{A7}$$

and Eqs. (2.11) and (2.12) for F_c and $F_*(-1)$. Here c_2, c_{10}, γ, and l_0 are constants, defined in (2.12), (3.8), and (5.16) above. As we should expect, for the terms that are $\propto 1/n$ the coefficients coincide with the corresponding coefficients given in (3.8).

The asymptotic forms (2.11) and (A7) are in good agreement with the results of numerical calculations even for $n\geqslant 3$, which is apparent from Figs. 1 and 4.

APPENDIX B

We discuss the width of the resonances for $E_r=U_m$ for a one-dimensional potential $U(x)$ with a barrier, a problem of interest in its own right. Assuming the potential is smooth, we apply quantization rules that take into account the finite permeability of the barrier.[10-12] For levels that are close to the top of the barrier, $U(x)$ can be approximated by the potential of an "inverted" (or repulsive)[43] oscillator:

$$U(x)\approx U_m-\frac{1}{2}\omega^2(x-x_m)^2,\quad \omega=[-U''(x_m)]^{1/2}. \tag{B1}$$

It will be clear from what follows that the parameter $a\propto 1/\log n\to 0$ for $n\gg 1$, which allows us to use the expansion

$$J(a)=\tilde{J}+\frac{1}{2\pi}a\ln a-J_1a+J_2a^2+\ldots, \tag{B2}$$

$$a=I+c_2I^2+c_3I^3+\ldots,\quad I=(U_m-E)/\omega, \tag{B3}$$

and (A1) for the function $\varphi(a)$ in (5.1). Here $J=1/\pi\int_{x_0}^{x_1}p(x,E)dx$ the quantization integral,

$$\tilde{J}=\frac{1}{\pi}\int_{\tilde{x}_0}^{x_m}\tilde{p}dx=n_0+1/2, \tag{B4}$$

$$J_1=\frac{1}{2\pi}\{\ln(n_0+1/2)+1+\ln[2\omega(x_m-\tilde{x}_0)/\tilde{J}]\}+\frac{1}{\pi}\int_{\tilde{x}_0}^{x_m}dx\left[\frac{\omega}{\tilde{p}(x)}-\frac{1}{x_m-x}\right], \tag{B5}$$

x_m is the point of maximum potential, $U_m=U(x_m)$; we denote quantities that are related to the (real) energy $E=U_m$ by an overbar, i.e., at the instant the level touches the top of the barrier: $\tilde{x}_1=x_m$, $\tilde{p}=p(x, E=O_m)$, etc. We note that J_1 is the regularization of the integral

$$\frac{1}{\pi}\int_{x_0}^{x_1}\omega p^{-1}dx=\frac{1}{2\pi}\omega T,$$

which diverges logarithmically as $E\to U_m$ (here T is the period of oscillation of a classical particle between the turning points x_0 and x_1). Finally, c_k in (B3) are coefficients that determine the value of the anharmonic corrections in the expansion of $U(x)$ when $x\approx x_m$, see Eq. (2.17) in Ref. 11 (not to be confused with the coefficients c_k in this article).

By substituting the expansion given above into the Eq. (1.3) from Ref. 11, it is not difficult to verify that terms $\propto a\ln a$ cancel, i.e., $a=0$ is not a singular point. Taking into account that at the instant the level touches the top of the barrier $I=i\delta/2$ ($\delta=\Gamma/\omega$), we are led to equations that determine δ and the shift in the resonance $\Delta n=n-n_0$ (due to the finite permeability of the barrier):

$$[\ln(n_0+1/2)+h_1]\delta+h_2\delta^2+h_3\delta^3+\cdots=\ln 2,$$

$$\Delta n=k_1\delta+k_2\delta^2+O(\delta^3), \tag{B6}$$

where

$$h_1=C+\ln[8\omega(x_m-\tilde{x}_0)^2/\tilde{J}]+2\int_{\tilde{x}_0}^{x_m}dx\left(\frac{\omega}{\tilde{p}}-\frac{1}{x_m-x}\right),$$

$$h_2=\frac{\pi}{4}\left(\frac{\pi}{2}-c_2\right),\quad k_1=\frac{1}{8},$$

$$k_2=\frac{1}{8\pi}\{[\ln(n_0+1/2)+h_1]c_2-2\pi J_2\},\ldots. \tag{B7}$$

From this we find in the logarithmic approximation ($\ln(n+1/2)\gg 1$):

$$\Gamma_n=\frac{\omega\ln 2}{\Lambda}[1+O(\Lambda^{-3})],$$

$$\Delta n=\frac{\ln 2}{8\Lambda}\left[1+\frac{c_2\ln 2}{\pi}+\cdots\right], \tag{B8}$$

where $\Lambda=\ln(n+1/2)+h_1+h_2\ln 2$.

Let us illustrate these expressions with the example of the potential (8.5). In this case

$$r_m=(g/\mathscr{E})^{1/(\alpha+1)},\quad U_m=-\alpha^{-1}\omega^2r_m^2,$$

$$\omega=(\alpha+1)^{1/2}(\mathscr{E}^{\alpha+2}/g)^{1/2(\alpha+1)}. \tag{B9}$$

The semiclassical momentum corresponding to an energy $E=U_m$ equals

$$\bar{p}=\omega r_m\left\{\frac{2}{\alpha(\alpha+1)}[x^{-\alpha}-1-\alpha(1-x)]^{1/2},\quad x=r/r_m,\right.$$

from which we have

$$\bar{J}=A_\alpha g^{3/2(\alpha+1)}\mathscr{E}^{-(2-\alpha)/2(\alpha+1)}, \tag{B10}$$

$$A_\alpha=(2/\alpha)^{1/2}\pi^{-1}\int_0^1[x^{-\alpha}-1-\alpha(1-x)]^{1/2}dx,$$

$$h_1=C+\frac{1}{2}\ln(\alpha+1)-\ln\frac{A_\alpha}{8}-2\int_0^1\varphi(x)dx, \tag{B11}$$

$$\varphi(x)=(1-x)^{-1}-\left\{\frac{\alpha(\alpha+1)}{2[x^{-\alpha}-1-\alpha(1-x)]}\right\}^{1/2} \tag{B12}$$

(as $x\to 1$ the pole singularities in $\varphi(x)$ cancel completely: $\varphi(1)=(\alpha+2)/6$; therefore, the integral in (B11) converges for all α). To summarize, we have

$$\omega_n=(\alpha+1)^{1/2}(A_\alpha)^{(2+\alpha)/(2-\alpha)}g^{2/(2-\alpha)}n^{-(2+\alpha)/(2+\alpha)}, \tag{B13}$$

$$\bar{\Gamma}_n\approx\gamma n^{-(2+\alpha)/(2-\alpha)}(\ln n+l_0)^{-1},\quad n\gg 1, \tag{B14}$$

where

$$\gamma=(\alpha+1)^{1/2}(A_\alpha)^{(2+\alpha)/(2-\alpha)}\ln 2,\quad l_0=h_1. \tag{B15}$$

From (B14) it is clear that resonances with $n\gg 1$ are narrow for $E\approx U_m$; their widths decrease with increasing n primarily according to a power law (the exponent of the power law is determined by the behavior of the potential at small distances). For $\alpha=1$, which corresponds to a Coulomb singularity $V(r)$ at zero, the asymptotic form (B14) coincides functionally with (5.7), while $\gamma\sim Z^2$ [the coupling constant $g=Z$ in (9.5)].

In conclusion, let us discuss a few special cases.

1) $\alpha=0$. Going from general to specific in (B11), we obtain

$$A_0=\frac{\sqrt{2}}{\pi}\int_0^1(x-\ln x-1)^{1/2}\,dx=0.2468,$$

$$\gamma=A_0\ln 2=0.17106,\quad l_0=3.2320. \tag{B16}$$

2) For $\alpha=1/2$ the integral can be computed in terms of elementary functions:

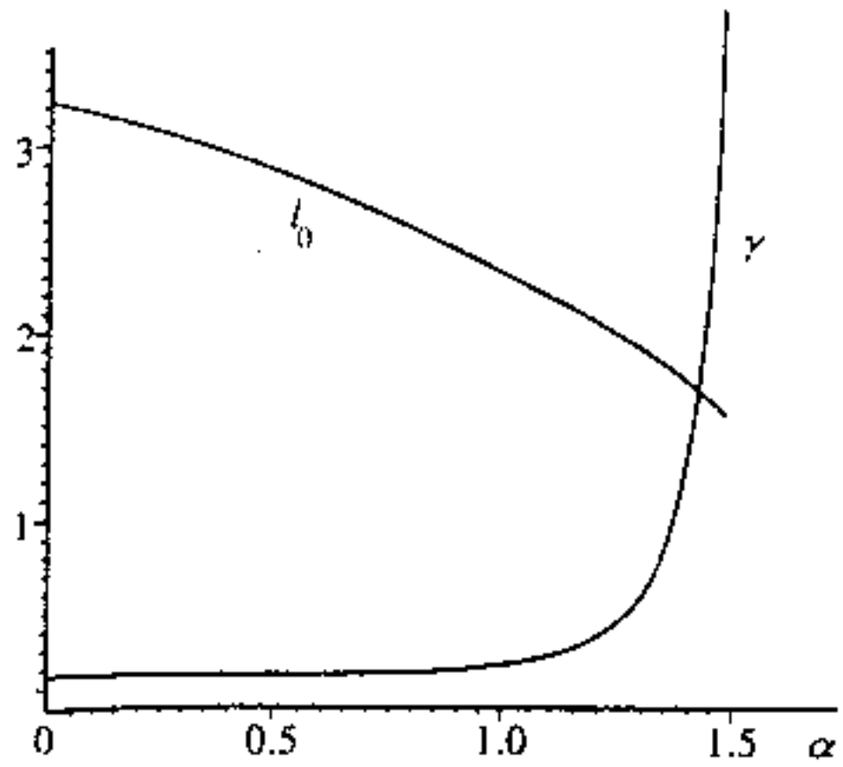

FIG. 10. Parameters of the asymptotic form (5.7) for the spherical model (9.5).

$$A_{1/2}=\frac{2^{3/2}}{\pi}(\sqrt{3}-\text{Ar sinh}\,\sqrt{3})=0.3737,$$

$$\int_0^1\varphi(x)dx=3-\ln 12,$$

from which

$$\gamma=(A_{1/2})^{5/3}\left(\frac{3}{2}\right)^{1/2}\ln 2=0.16460,\quad l_0=2.8135 \tag{B17}$$

(the constant γ in (B15) has a minimum for $\alpha=1/2$).

3) $\alpha=1$. Just as in the previous case, the computations can all be done analytically:

$$A_1=\frac{2^{5/2}}{3\pi}=0.6002,\quad \varphi(x)=(1+x^{1/2})^{-1},$$

$$\int_0^1\varphi dx=2(1-\ln 2),$$

which gives

$$\gamma=A_1^3\cdot 2^{1/2}\ln 2=\frac{2^8}{(3\pi)^3}\ln 2=0.21196,$$

$$l_0=C+\ln 96\pi-4=2.2863. \tag{B18}$$

We note that the results in (B18) coincide exactly with the constants in the asymptotic form (5.7) for the Rydberg states $(0,n-1,0)$ in the three-dimensional problem. This is explained by the fact that the quasiclassical equations for the energies of these states are the same as for the "spherical" model of the Stark effect, which is described by the potential (8.5) with $\alpha=1$. For this case it is not difficult to calculate the coefficients of the expansion (B3) as well:

$$c_2=\frac{1}{4\pi(n+1/2)},\quad c_3=-\frac{5}{144\pi^2(n+1/2)^2}. \tag{B19}$$

Hence it is clear that c_k are small for states with large n and can be omitted in (B7).

The results of numerical calculations using Eqs. (B11), (B15) are shown in Fig. 10, from which it is clear that γ is a "sluggish" function of the exponent α in the range $0<\alpha\lesssim 1.3$ (however, γ increases rapidly for $\alpha>1.5$; note that the value $\alpha=2$ corresponds to the "fall to the center" of Ref. 13). On the other hand, the constant l_0 decreases monotonically with increasing α, while $l_0>2$. Therefore, the term with $\ln n$ in the asymptotic form (B14), which dominates as $n\to\infty$, is not dominant for values of $n\sim 20$.

APPENDIX C

We discuss the calculation of the integrals σ_i,τ_j from Sec. 5 for states with $m=0$. In this case the centrifugal term in $k_i^2(q)$ vanishes, and the turning points can be found in explicit form:

$$k_1=\frac{1}{2}F^{1/2}\left[\frac{(x_1-x)(x-x_2)}{x}\right]^{1/2},$$

$$k_2=\frac{1}{2}F^{1/2}\left[\frac{(y-y_1)(y-y_2)}{y}\right]^{1/2},$$

where $x_0=y_0=0$,

$$x_1=\frac{\epsilon}{2F}[1-(1+z_1)^{1/2}],\quad x_2=\frac{\epsilon}{2F}[1+(1+z_1)^{1/2}, \tag{C1}$$

$$y_{1,2}=-\frac{\epsilon}{2F}[1\mp(1-z_2)^{1/2}],\quad z_i=\frac{16\beta_i F}{\epsilon^2} \tag{C2}$$

($\epsilon<0$, therefore, $x_1>0$, while $x_2<0$). At the instant the levels touch the top of the barrier we have: $z_1=z(\kappa)$, $z_2=1$,

$$\bar{k}_2(y)=\omega(y/y_m)^{-1/2}(y_m-y),\quad \omega=\tfrac{1}{2}(F/y_m)^{1/2}. \tag{C3}$$

Using the integral representation (2.12.1) from the handbook Ref. 45, we can express J_1 and σ_i in terms of hypergeometric functions of the argument $w=[1-(1+z_1)^{1/2}]/[1+(1+z_2)^{1/2}]$. If we then make use of the Kummer transformation[45]

$$F\left(\alpha,\alpha-1;2;\frac{1-(1+z)^{1/2}}{1+(1+z)^{1/2}}\right)$$

$$=\left[\frac{1+(1+z)^{1/2}}{2}\right]^{\alpha}F\left(\frac{\alpha}{2},\frac{\alpha+1}{2};2;-z\right),$$

we finally obtain

$$J_1=\beta_1(-\epsilon)^{-1/2}F(1/4,3/4;2;-z_1),$$

$$\sigma_0=2\pi\beta_1(-\epsilon)^{-3/2}F(3/4,5/4;2;-z_1), \tag{C4}$$

$$\sigma_1=\pi(-\epsilon)^{-1/2}F(1/4,;3/4;1;-z_1),$$

$$s=\frac{\sigma_0}{\sigma_1}=-\frac{2\beta_1}{\epsilon}\,\frac{F(3/4;5/4;2;-z_1)}{F(1/4,3/4;1;-z_1)}. \tag{C5}$$

Here assuming $z_1=z$ and taking (3.7) into account, we are led after several transformations to Eq. (5.12).

We note that the integrals σ_i are easily calculated for $\kappa=1$, where $x_1=-x_2=9\pi/8K$,

$$k_1(x)=\tfrac{1}{2}F_*^{1/2}[(x_1^2-x^2)/x]^{1/2},$$

$$\sigma_i=\left(\frac{F_*}{\pi}\right)^{-1/2}\frac{\Gamma\left(\frac{3}{4}-\frac{j}{2}\right)}{\Gamma\left(\frac{5}{4}-\frac{j}{2}\right)}x_1^{1/2-j}, \tag{C6}$$

from which $s(1)=\pi(3/4K)^2$. This result also follows from (5.12) and (3.6), if we take into account that

$$F(3/4,3/4;2;1)/F(1/4,1/4;1;1)=2K^{-1},$$

where K is the constant entering into (3.5).

Taking (C3) into account, calculating the integrals τ_j from (5.10) is elementary:

$$\tau_j=a_j/y_m^j, \tag{C7}$$

$$a_j=-\int_0^1\frac{1-t^{1/2-j}}{1-t}\,dt=\psi(1)-\psi\left(\frac{3}{2}-j\right), \tag{C8}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. In particular,

$$a_0=-2(1-\ln 2),\quad a_{1/2}=0,\quad a_1=2\ln 2, \tag{C9}$$

which gives Eq. (5.13). Since Eq. (5.7) is asymptotic, in the computations the constants that enter into it can be taken to the Rydberg limit $n\to\infty$, i.e., we can replace F_c with F_*, etc., which was used in the derivations above.

[1)]From physical considerations we choose that root y_m which goes to infinity as the electric field is turned off, i.e., the root $y_m\approx 2/3F$ as $F\to 0$.

[2)]In what follows, we denote the coefficients of expansions encountered below by c_k; their numerical values are given in Table III. Although all coefficients c_k can be calculated in analytic form, we omit certain of these expressions because of their unwieldiness

[3)]See Refs. 9, 32. As $n\to\infty$, we can omit the function $\varphi(a)$ in the second of these equations, i.e., we can consider the barrier impenetrable (for $F<F_*$)

[4)]As in the case of the $1/n$ expansion,[16,19] this leads to the circumstance that for $F>F_*$ the classical solution begins to describe not only the position but also the width of the resonance. We refer to the solution to Eqs. (3.1) as "classical" because these equations are the limit (as $n\to\infty$) of the semiclassical quantization rules.

[5)]Except for the circular states $(0,0,n-1)$ with $n\gg 1$, for which the $1/n$ expansion rather rapidly leads to Eqs. (2.10).

[6)]These expressions are conveniently derived for the general case of systems with $f\geq 2$ degrees of freedom that allow separation of variables in the Schrödinger equation. This question requires a separate investigation. The basic idea of the derivation is set forth in Appendix B for the technically simpler example of a one-dimensional potential with a barrier.

[7)]For the other states, the energy E_r can also intersect the boundary $E=0$, but the corresponding fields are too large. Thus, in the case of the ground state $F_0\approx 2.3$ a.u. (see Fig. 8 below).

[1]T. S. Luk, L. Di Mauro, T. Bergeman *et al.*, Phys. Rev. Lett. **47**, 83 (1981).
[2]W. L. Glab and M. H. Nayteh, Phys. Rev. A **31**, 530 (1985).
[3]K. Ng, D. Yao, and M. H. Nayteh, Phys. Rev. A **35**, 2508 (1987).
[4]S. Augst, D. Strickland, D. D. Meyerhofer *et al.*, Phys. Rev. Lett. **63**, 2212 (1989).
[5]G. Gibson, T. S. Luk, and C. K. Rhodes, Phys. Rev. A **41**, 5049 (1990).
[6]M. Dörr, R. M. Potvliege, and R. Shakeshaft, Phys. Rev. Lett. **64**, 2003 (1990); R. Shakeshaft, R. M. Potvliege, and M. Dörr, Phys. Rev. A **42**, 1656 (1990).
[7]A. Nussenzweig, E. E. Eyler, T. Bergeman *et al.*, Phys. Rev. A **47**, 4944 (1990).
[8]V. V. Kolosov, JETP Lett. **44**, 588 (1986).
[9]V. M. Weinberg, V. D. Mur, V. S. Popov *et al.*, JETP Lett. **44**, 9 (1986); JETP Lett. **46**, 225 (1987); Zh. Eksp. Teor. Fiz. **93**, 450 (1987) [Sov. Phys. JETP **66**, 258 (1987)].
[10]V. D. Mur and V. S. Popov *et al.*, JETP Lett. **51**, 563 (1990)
[11]V. S. Popov, V. D. Mur, and A. V. Sergeev, Zh. Eksp. Teor. Fiz. **100**, 200 (1991) [Sov. Phys. JETP **73**, ?? (1991)]; Phys. Lett. A **157**, 185 (1991).
[12]J. N. L. Connor, Mol. Phys. **25**, 1469 (1973).
[13]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (Nonrelativistic Theory)*. Nauka, Moscow (1974); Pergamon Press, Oxford (1977).
[14]S. Yu. Slavyanov, Problemy Mat. Fiz. No. 4, 125 (1970).
[15]T. Yamabe, A. Tachibana, and H. J. Silverstone, Phys. Rev. A **16**, 877 (1977).
[16]V. S. Popov, V. D. Mur, and V. M. Weinberg, Phys. Lett. A **124**, 77 (1987).
[17]H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Systems*, Hand. Phys. **35**, 88. Springer, Berlin (1957).
[18]C. M. Bender, L. D. Mlodinow, and N. Papanicolaou, Phys. Rev. A **25**, 1305 (1982).
[19]V. S. Popov, V. M. Weinberg, and V. D. Mur, JETP Lett. **41**, 539 (1985).
[20]S. Kais and D. R. Herschbach, J. Chem. Phys. **98**, 3990 (1993).
[21]V. D. Mur, V. S. Popov, and A. V. Sergeev, Zh. Eksp. Teor. Fiz. **97**, 32 (1990) [Sov. Phys. JETP **70**, 16 (1990)].
[22]V. M. Weinberg, V. S. Popov, and A. V. Sergeev, Zh. Eksp. Teor. Fiz. **98**, 847 (1990) [Sov. Phys. JETP **71**, 470 (1009)].
[23]V. S. Popov, V. D. Mur, and A. V. Sergeev, JETP Lett. **59**, 158 (1994).
[24]V. S. Popov, Preprint ITEP 58-92 [in Russian], Moscow (1992); Phys. Lett. A **173**, 63 (1993).
[25]R. E. Langer, Phys. Rev. **51**, 669 (1937).
[26]J. Heading, *Introduction to Phase Integral Methods (The WKB Method)*, Wiley, New York (1962).
[27]A. B. Migdal, *Qualitative Methods in Quantum Theory* [in Russian], Nauka, Moscow (1975); *Frontiers in Physics* Series, vol. 48, Benjamin-Cummings (1977).
[28]V. S. Popov, V. D. Mur, A. V. Sergeev *et al.*, preprint IC/89/320, Trieste (1989); Phys. Lett. A **149**, 418 (1990); Phys. Lett. A **149**, 425 (1990).
[29]V. D. Mur and V. S. Popov, JETP Lett. **57**, 418 (1993); Zh. Eksp. Teor. Fiz. **104**, 2293 (1993) [Sov. Phys. JETP **77**, 18 (1993)].
[30]M. B. Kadomtsev and B. M. Smirnov, Zh. Eksp. Teor. Fiz. **80**, 1715 (1981) [Sov. Phys. JETP **53**, 885 (1981)].
[31]V. S. Popov, V. D. Mur, and A. V. Sergeev, Preprint ITEP 173-88 [in Russian], Moscow (1988).
[32]V. D. Mur, V. S. Popov, and A. V. Sergeev, Zh. Eksp. Teor. Fiz. **96**, 91 (1989) [Sov. Phys. JETP **69**, 49 (1989)].
[33]V. P. Krainov, in: *Proc. VI Intl. Conf. on Multiphoton Processes*, Laval Univ. Publ., Quebec (1993).
[34]V. D. Kondratovich and V. N. Ostrovsky, J. Phys. B **17**, 1981 (1984); J. Phys. B **17**, 2011 (1984).
[35]G. E. Gorelik, *The Dimensionality of Space* [in Russian], Moscow State Univ., Moscow (1983).
[36]D. R. Herschbach *et al.* (eds.), *Dimensional Scaling in Chemical Physics*, Kluwer Acad. Publ., Dordrecht, Netherlands (1993)
[37]X. L. Yang, S. H. Quo, F. T. Chan *et al.*, Phys. Rev. A **43**, 1186 (1991).
[38]H. J. Silverstone, Phys. Rev. A **18**, 1853 (1978).
[39]S. P. Alliluev, V. L. Eletsky, and V. S. Popov, Phys. Rev. A **73**, 103 (1979); Phys. Rev. A **78**, 43 (1980).
[40]V. Prizman, Phys. Rev. A **22**, 1833 (1980).
[41]A. D. Dolgov and V. S. Popov, Zh. Eksp. Teor. Fiz. **75**, 2010 (1978) [Sov. Phys. JETP **48**, 1012 (1978)]; Phys. Rev. B **79**, 403 (1978); Phys. Rev. B **86**, 185 (1979).
[42]Y. Aharonov and C. K. Au, Phys. Rev. Lett. **45**, 1582 (1979); Phys. Rev. A **20**, 2245 (1979).
[43]S. P. Alliluev and V. S. Popov, Zh. Eksp. Teor. Fiz. **104**, 3569 (1993) [Sov. Phys. JETP **77**, 701 (1993)].
[44]W. Miller Jr., *Symmetry and Separation of Variables*, Addison-Wesley, Reading, Mass. (1977).
[45]H. Bateman, *Higher Transcendental Functions* Vol. 1. McGraw-Hill, New York (1953).

Translated by Frank J. Crowne