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Theory

Coulomb Corrections to Scattering Lengths and Effective Ranges for $l \neq 0$

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Abstract—Coulomb corrections to the low-energy scattering parameters (scattering lengths and effective ranges) of states with nonzero orbital angular momentum are studied. The greatest attention is paid to the case of the p wave. Resonance scattering is considered for a system with a shallow nuclear bound or quasistationary (resonance) p level. Coulomb corrections are calculated numerically, and their dependence on the shape of the strong-interaction potential—in particular, on its short-distance behavior—is studied in detail.

1. INTRODUCTION

This article reports a continuation of the study begun in [1]. Here, we investigate Coulomb corrections for states with $l \neq 0$, paying special attention to the case of the p wave. The formulation of the problem is given in [1], where the reader can also find an extensive list of references on the subject. In view of this, we skip introductory remarks and begin by describing the organization of the paper.

In Section 2, we present the formulas for Coulomb corrections to the parameters of low-energy scattering (scattering length $a_l^{(s)}$ and effective range $r_l^{(s)}$ associated with strong interaction) in states with arbitrary orbital angular momentum l. As before, we consider the case of resonance scattering (without absorption) for a system having a shallow (on the nuclear scale) bound l-wave level, so that the above corrections may be very important. The cases of $l = 0$ and $l \geq 1$ are different. Indeed, for the lth partial wave, the large Coulomb logarithm

$$\Lambda = \ln(1/\delta), \quad \delta \sim r_N/a_{\rm B} \ll 1, \qquad (1)$$

appears only in the Coulomb-renormalized coefficient of k^{2l} in the effective-range expansion (see [2, 3]). For this reason, the quantity that undergoes the strongest renormalization in the s wave is the scattering length [4], the term proportional to Λ appears in the renormalized effective range $r_1^{(s)}$ for the p wave ($l = 1$), etc. Coulomb corrections to the remaining terms of the effective-range expansion involve additional powers of the small parameter δ (we assume that $|\delta| \ll 1$; this is so at least for extremely light hadronic systems). Thus, the cases of $l = 0$ (see [1]) and $l = 1$, which are of paramount importance for applications, must be considered separately.

In Section 3, we calculate numerically the coefficients f_1 and h_1 [see formulas (12) below] for various model potentials of strong interaction. These coefficients determine the Coulomb corrections to the quantities $a_1^{(s)}$ and $r_1^{(s)}$. We also consider the dependence of the Coulomb corrections on the form of strong-interaction potential—in particular, on its small-distances behavior. Some formulas and details of calculations are given in the Appendices.

In what follows, we use the system of units in which $\hbar = m = e = 1$, where m is the reduced mass, and the conventions adopted in [1].

2. COULOMB CORRECTIONS FOR LOW-ENERGY SCATTERING

For states with $l \neq 0$, the Coulomb renormalization of the inverse scattering length is given by [5]

$$\frac{1}{a_l^{(cs)}} - \frac{1}{a_l^{(s)}} = -c_l \frac{\sigma}{a_{\rm B}} J_l, \qquad (2)$$

where $c_l = 2[(2l-1)!!]^2$, $\sigma = \mathrm{sgn}(Z_1 Z_2) = \pm 1$, $a_{\rm B}$ is the Bohr radius, and

$$J_l = \int_0^\infty \chi_l^2(r) \frac{dr}{r}. \qquad (3)$$

Here, $\chi_l(r)$ is the wave function in the strong-interaction potential $V_s(r)$ corresponding to the emergence of an l-wave level. This function satisfies the Schrödinger

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equation with energy $E = 0$ and the boundary conditions

$$\chi_l(r) \propto r^{l+1}, \quad r \longrightarrow 0; \quad \lim_{r \to \infty} r^l \chi_l(r) = 1. \qquad (4)$$

For nonzero l values, the function $\chi_l(r)$ decreases at infinity (owing to the centrifugal barrier) and is normalizable; that is, we have

$$N_l = \int_0^\infty \chi_l^2(r)dr < \infty. \qquad (5)$$

The effective range is negative; it is given by [6]

$$r_l^{(s)} = -c_l N_l, \quad l \geq 1, \qquad (6)$$

and can be estimated as $\left|r_l^{(s)}\right| \sim r_N^{1-2l}$, where r_N is the range of nuclear forces. It is convenient to express the right-hand side of equation (2) in terms of the effective range by using the relation

$$\frac{1}{a_l^{(s)}} - \frac{1}{a_l^{(cs)}} = d_l \frac{\sigma}{a_B}\left|r_l^{(s)}\right|^{2l/(2l-1)}, \qquad (7)$$

where

$$d_l = c_l^{-1/(2l-1)}(N_l)^{-2l/(2l-1)} J_l \qquad (8)$$

is the dimensionless coefficient depending on the form of the potential $V_s(r)$ and on the number of the l-wave level. Recall that

$$\tilde{a}_1 = \frac{1}{3}R^3, \quad \tilde{r}_1 = -3.6R^{-1} \quad (l = 1) \qquad (9)$$

for a hard sphere of radius R (nonresonance case)[3] and

$$a_1 = \infty, \quad r_1^{(s)} = -3R^{-1} \qquad (10)$$

for a square well of the same radius ($g = g_{nl}$; see Appendix 1).

For an arbitrary short-range potential $V_s(r)$, it is convenient to perform normalization by using (10) because we assume that the condition $|a_1| \gg r_N^3$ is satisfied (quasiresonance case). Introducing the parameter R_1 (having dimensions of length) via the relation

$$R_1 = -3/r_1^{(s)}, \quad \delta = R_1/a_B, \qquad (11)$$

we can represent the Coulomb corrections for the case of the p wave as

$$\frac{1}{a_1^{(s)}} - \frac{1}{a_1^{(cs)}} = \frac{\sigma}{\tilde{a}_1} f_1 \delta + \ldots, \quad f_1 = 3d_1 = 1.5J_1/N_1^2,$$
$$r_1^{(s)} - r_1^{(cs)} = -4\frac{\sigma}{a_B}[\ln(1/\delta) + h_1] + \ldots, \qquad (12)$$

[3] In what follows, tilde-labeled quantities correspond to the case of a hard sphere.

Table 1. Ratios R_l/r_s for potentials of the form (15)

$v(x)$	$l = 1$	$l = 2$	$l = 3$	$l = 5$	$l = \infty$
$e^{-x}x^{-1}$	1.030	1.034	1.033	1.027	0.979
$(e^x - 1)^{-1}$	1.041	1.053	1.056	1.057	1.020
$1/\sinh x$	1.037	1.048	1.051	1.051	1.016
$\exp(-x^2)x^{-1}$	1.050	1.070	1.080	1.088	1.072
$e^{-x}(1 + x)$	1.013	1.016	1.016	1.013	1.062
$e^{-x}(1 + x)^{-1}$	1.012	1.013	1.010	1.004	0.958
$(e^x + 1)^{-1}$	1.012	1.014	1.013	1.009	0.968
$x/(e^x - 1)$	1.017	1.022	1.023	1.021	0.986
$(\cosh x)^{-2}$	1.0077	1.0080	1.0062	1.0016	0.960
$\exp(-x)$	1.015	1.018	1.018	1.014	0.973
$\exp(-x^2)$	1.015	1.022	1.025	1.027	1.004
$\exp(-x^4)$	1.007	1.011	1.013	1.016	1.002
$\theta(1 - x)$	1	1	1	1	1

Note: The values of R_l (13) correspond to the emergence of the first (nodeless) l-wave level; r_s is the effective range for the ground state with ($l = 0$) (see Table 1 from [1]).

where the ellipses stand for terms involving additional smallness in $\delta \sim r_N/a_B$. The explicit expression for the coefficient h_1 was obtained in [2] and is presented in Appendix 2 (in a more convenient form). As was mentioned above, the Coulomb logarithm (1) appears in the correction to the effective range (not in correction to the scattering length, as in the case of $l = 0$).

Thus, the Coulomb-renormalized inverse scattering length and effective range at $l = 1$ are expressed in terms of the radius R_1 (or $r_1^{(s)}$) and dimensionless coefficients f_1 and h_1. A method for calculating these coefficients is described in Appendix 2.

The above parameter R_1 can be generalized to the case of arbitrary l as

$$R_l = \begin{cases} r_0^{(s)} \equiv r_s, & l = 0 \\ \left[\frac{(2l+1)c_l}{2(2l-1)} / \left|r_l^{(s)}\right|\right]^{\frac{1}{2l-1}}, & l \geq 1, \end{cases} \qquad (13)$$

where c_l is the same numerical coefficient as in (2). For a square-well potential, the parameter R_l coincides with the well radius (that is, with r_s) at all l and n values (see Appendix 1). It can be seen from Table 1 that, for other short-range potentials, R_l for $l \leq 3$ differs from the effective range r_s for the ground state only by several percent. At the same time, the effective ranges essentially depend on the form of the strong-interaction potential and increase factorially with increasing l (see Appendix 1).

Table 1 also presents the result obtained in the limit $l = \infty$, which can easily be treated within the technique of $1/n$ expansion. Using the asymptotic expressions for the normalization integral N_l {see formula (4.3) from [7]}, we obtain

$$R_l/r_N \approx x_0 \exp\left\{\int_{x_0}^{\infty}\frac{dx}{x}\left[1-\sqrt{1-\frac{x^2 v(x)}{x_0^2 v(x_0)}}\right]\right\}, \quad l \longrightarrow \infty, \qquad (13a)$$

where x_0 is the position of the turning point for $E = 0$ and $l \longrightarrow \infty$, which is determined from the equation $xv'(x) + 2v(x) = 0$ with the potential used in the form (15).

Table 1 demonstrates that, for short-range potentials, the quantities R_l and r_s are close to each other for all $l \geq 1$. This indicates that the parameter R_l introduced in (13), which has dimensions of length for any l value and which determines the effective size of the system, has been chosen reasonably. Note also that R_l approaches the limiting value R_∞ rather slowly. This can be seen from Table 1 and the model example employing the δ potential that admits construction of analytic solutions. In the latter case, equation (A.10) yields

$$R_l/R_\infty = 1 + \frac{\ln(l/2)}{2l} + O\left(\frac{\ln l}{l^2}\right), \quad l \longrightarrow \infty.$$

Thus, the expansion parameter δ (11) is very close to the parameter $\delta = r_s/a_B$ used in [1] for the case of s states. The analogs of formulas (12) for an arbitrary orbital angular momentum $l \geq 1$ are given by

$$\frac{1}{a_l^{(s)}} - \frac{1}{a_l^{(cs)}} = \frac{\sigma}{\tilde{a}_l} f_l \delta + \ldots, \quad \delta = R_l/a_B,$$
$$r_l^{(s)}/r_l^{(cs)} = 1 - h_l\delta + O(\delta^2), \quad l \neq 1, \qquad (14)$$

where

$$\tilde{a}_l = \alpha_l R_l^{2l+1} = \alpha_l'\left|r_l^{(s)}\right|^{-(2l+1)/(2l-1)} \qquad (14a)$$

is the scattering length for interaction with a hard sphere of radius R_l and

$$f_l = \left(l+\frac{1}{2}\right)^{\frac{1}{2l-1}}[(2l-1)N_l]^{-2l/(2l-1)} J_l$$
$$= [(2l+1)!!(2l-3)!!]^{\frac{1}{2l-1}}(2l-1)^{-1} d_l. \qquad (14b)$$

The coefficients α_l are defined in (A.7); $\alpha_1' = 9$, $\alpha_2' = 2.097$, $\alpha_3' = 1.997$, $\alpha_4' = 2.193$, and α_l' for $l \gg 1$ is approximately equal to $0.271l$. The dimensions of the quantities introduced above are

$$[\chi_l] = L^{-l}, \quad [J_l] = L^{-2l}, \quad [N_l] = L^{1-2l},$$
$$[a_l] = L^{2l+1}, \quad [r_l] = L^{1-2l}, \quad [R_l] = L,$$

where L has dimensions of length. In particular, we have $[a_0] = [r_0] = L$ for the s wave and $[a_1] = L^3$ and $[r_1] = L^{-1}$ for the p wave.

Formulas (12)–(14) express the Coulomb corrections to the parameters of low-energy scattering in terms of the effective range of the system for the interaction strength corresponding to the emergence of the l-wave state. In principle, this effective range can be measured in experiments. In addition, these formulas contain the dimensionless coefficients f_l and h_l, which depend on the form of the strong-interaction potential and which are determined numerically (see Appendix 2). We now describe the results of our calculations.

3. NUMERICAL CALCULATIONS

A short-range strong-interaction potential can be chosen as

$$V_s(r) = -\frac{g}{2r_N^2}v(x), \quad x = r/r_N, \qquad (15)$$

where the function $v(x)$ determines its form, and g is the dimensionless coupling constant. The results of our calculations are presented in Table 2, where $x_1^s = -r_1^s r_N$ is given by (A.26). All the quantities in Table 2 for all potentials, with the exception of the Yukawa potential (no. 1), correspond to the emergence of the first $l = 1$ bound state ($g = g_1$) in the potential (15).

Table 2 lists the results for potentials having a Coulomb singularity at the origin (nos. 1–9), including the Yukawa and Hulthén (no. 5) potentials, and for various potentials that are finite at the origin. The following conclusions can be drawn from analysis of these data.

(1) First, we emphasize that the coefficient f_1 is almost constant, lying between 0.9 and 1.0 for the majority of the potentials considered here; it varies only slightly even when we consider excited states. Thus, the Coulomb corrections to the scattering length $a_1^{(s)}$ are weakly dependent on the specific form of the strong-interaction potential and seem to be model-independent quantities.

(2) On the other hand, the coefficient h_1 changes strongly and irregularly when we go over from one form of potential to another. To determine the Coulomb correction to the effective range $r_1^{(s)}$, we therefore need detailed information about the potential $V_s(r)$.[4)]

Previously, the coefficient h_1 was calculated only for a δ-function and a square-well potential. In these cases, which can be treated analytically, the resulting h_1 values are close to each other. On this basis, it was assumed in [5] that the coefficient h_1 takes close values for more realistic potentials as well. Data from Table 2 demonstrate that this is not the case.

[4)] It should be noted that the coefficient h_1 is added in (12) to the large Coulomb logarithm. As a result, the Coulomb correction to the effective range proves to be less sensitive to the form of the potential $V_s(r)$.

Table 2. Parameters of short-range attractive potentials ($l = 1$)

No.	$v(x)$	g_1	x_1^s	x_1^C	f_1	h_1	R_1/r_s
1	$e^{-x}x^{-1}$	9.08196	1.374	0.209	1.024	1.497	1.030
		17.7446	0.952	0.670	0.981	0.700	0.795
		29.4614	0.781	0.989	0.947	0.509	0.747
		44.2613	0.683	1.226	0.923	0.429	0.729
		62.1602	0.619	1.413	0.906	0.385	0.721
2	$\exp(-x^2)x^{-1}$	8.79424	2.670	0.264	0.920	0.600	1.050
3	$\exp(-x^4)x^{-1}$	7.82073	3.251	0.265	0.872	0.400	1.051
4	$x^{-1}\theta(1-x)$	6.59365	3.273	0.287	0.840	0.313	1.051
5	$(e^x-1)^{-1}$	5.30594	0.960	0.449	0.988	1.092	1.041
6	$1/\sinh x$	3.28939	0.868	0.615	0.955	0.878	1.037
7	$(e^{2x}-e^x)^{-1}$	13.9487	2.201	0.153	1.013	1.338	1.035
8	$(2e^{-x}-e^{-2x})x^{-1}$	5.65691	1.234	0.315	0.990	1.195	1.029
9	$(x^{-1}-1)\theta(1-x)$	15.3962	4.482	0.186	0.891	0.431	1.069
10	$\exp(-x)$	7.04906	0.835	0.701	0.935	0.787	1.015
11	$\exp(-x^2)$	12.0993	2.060	0.408	0.870	0.426	1.015
12	$\exp(-x^4)$	12.2985	2.787	0.333	0.837	0.326	1.007
13	$\theta(1-x)$	9.86960	3.000	0.325	0.813	0.276	1.000
14	$(e^x+1)^{-1}$	8.21353	0.784	0.816	0.919	0.699	1.012
15	$1/\cosh x$	3.68891	0.815	0.752	0.925	0.741	1.008
16	$1/\cosh^2 x$	9.35909	1.489	0.457	0.907	0.636	1.008
17	$x/\sinh x$	1.53231	0.611	1.165	0.901	0.591	1.011
18	$(1-x^2)\theta(1-x)$	21.6796	3.599	0.263	0.831	0.306	1.012
19	$2e^{-x}-e^{-2x}$	3.87529	0.801	0.778	0.923	0.724	1.011
20	$x/(e^x-1)$	2.80068	0.635	1.069	0.913	0.638	1.017

Note: For the Yukawa potential, the parameters are presented for the five p-wave levels. For other potentials, they are given for the first $l = 1$ level. All the quantities correspond to the emergence of the p level, and the potentials are represented in the form (15).

(3) Assuming that there is a power-law singularity for $r \longrightarrow 0$, we now investigate the dependence of the Coulomb corrections on the short-distances behavior of the potential $V_s(r)$. For this purpose, we consider the functions

$$v(x) = x^{-\alpha}\exp(-x), \quad 0 \le \alpha < 2 \tag{16}$$

$$v(x) = x^{-\alpha}\theta(1-x), \tag{17}$$

which correspond to a smooth and a sharp cutoff of the strong-interaction potential at large distances. With increasing α, the x_1^s value, which determines the effective range, decreases. This is because the wave function tends to concentrate near the origin (it is well known [8] that collapse into the origin occurs at $\alpha = 2$). At the same time, the coefficient f_1 is almost constant, while the coefficient h_1 strongly increases with increasing α (compare the results for potentials nos. 1–4 and 10–13 in Table 2). This is in accord with the above conclusions.

(4) Comparison of the results for potentials nos. 1–4 and 10–13 from Table 2 indicates that the quantities $\left|r_1^{(s)}\right| = x_1^{(s)}/r_N$ increase, while the coefficient h_1 decreases as the decrease of the potential $V_s(r)$ at infinity becomes faster.

(5) For $l \geq 2$, the coefficients f_l appearing in (14) were also calculated and found to depend only slightly on the form of the potential $V_s(r)$. On the whole, they reveal a tendency to decrease with increasing l.

4. CONCLUSION

We have calculated Coulomb corrections to the characteristics of low energy scattering for various short-range potentials $V_s(r)$. The results obtained in [1] for $l = 0$ and here for $l = 1$ make it possible to find out which of these corrections are virtually independent of the choice of model and which essentially depend on the form of the strong-interaction potential. These results can be used in the theory of extremely light hadronic systems.

We emphasize that we always considered resonance scattering for a system with a shallow nuclear level. In the case of Coulomb attraction, this leads to rearrangement of the atomic spectrum (Zeldovich's effect).[5)] Such a situation does not occur frequently, because the relative width of the region where the spectrum is rearranged is determined by the small parameter $(r_N/a_B)^{2l+1}$ and decreases fast with increasing l. On the other hand, many resonance systems with Coulomb repulsion (pp, dt, d^3He, $\alpha\alpha$, etc.) are of great physical interest. Coulomb corrections for these systems are significant. It was shown above that, under the condition $a_B \gg r_N$, there are convenient formulas for calculating these corrections for any strong-interaction potential.

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APPENDIX 1

Let us consider several model potentials for which $r_l^{(s)}$ can be calculated analytically.

(1) The hard-sphere model is extensively used in kinetic theory of gases, molecular physics, plasma physics, etc. [12, 13]. The wave functions have the form $\chi_l(r) = \cos\delta_l J_\nu(kr) - \sin\delta_l N_\nu(kr)$ for $r > R$ and satisfy the boundary condition $\chi_l(r) = 0$. The effective-range function is given by [14, 15]

$$\tilde{K}_l^{(s)}(k^2) = k^{2l+1}\cot\delta_l(k) = -\frac{1}{\bar{a}_l}\lambda_l(kR), \quad \text{(A.1)}$$

where $\bar{a}_l$ is scattering length [see equation (A.6) below],

$$\lambda_l(z) = \Lambda_{-\nu}(z)/\Lambda_\nu(z), \quad \nu = l + 1/2, \quad \text{(A.2)}$$

and $\Lambda_\nu(z)$ is related to a Bessel function by the equation

$$\Lambda_\nu(z) = \Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu} J_\nu(z) \equiv {}_0F_1\left(\nu+1; -\frac{1}{4}z^2\right). \quad \text{(A.3)}$$

For the s and p waves, we have

$$\lambda_0(z) = z\cot z = 1 - \frac{1}{3}z^2 - \frac{1}{45}z^4 - \frac{2}{945}z^6 - \dots,$$
$$\lambda_1(z) = \frac{1}{3}z^3\left(\frac{1+z^2}{\tan z - z} + z\right) = 1 + \frac{3}{5}z^2 - \frac{12}{175}z^4 + \dots. \quad \text{(A.4)}$$

For arbitrary l and $z \longrightarrow 0$, we arrive at

$$\lambda_l(z) = 1 + \frac{2l+1}{2l+3} \times\left\{\frac{z^2}{2l-1} + \frac{l+3}{(2l+5)(2l+3)(2l-3)}z^4 + \dots\right\}. \quad \text{(A.5)}$$

From the above relations, it follows that, for a hard sphere of radius R and arbitrary l, the scattering length and effective range for the nonresonance case are given by

$$\bar{a}_l = \alpha_l R^{2l+1}, \quad \tilde{r}_l = -\tilde{\beta}_l R^{1-2l}, \quad R_l = \left(\frac{2l+3}{4l+2}\right)^{\frac{1}{2l-1}} R, \quad \text{(A.6)}$$

where

$$\alpha_l = [(2l+1)!!(2l-1)!!]^{-1}, \quad \tilde{\beta}_l = \frac{2(2l+1)}{(2l+3)(2l-1)\alpha_l}. \quad \text{(A.7)}$$

In particular, we have $\alpha_0 = 1$, $\tilde{\beta}_0 = -2/3$; $\alpha_1 = 1/3$, $\tilde{\beta}_1 = 18/5$; etc.

(2) For a square-well potential, the function appearing in (15) has the form $v(x) = \theta(1-x)$, $r_N \equiv R$. The scattering length is

$$a_l^{(s)} = \left[1 - \frac{\Lambda_\nu(\kappa)}{\Lambda_{\nu-1}(\kappa)}\right]\bar{a}_l = -\frac{\kappa^2}{(2l+1)(2l+3)}\frac{\Lambda_{\nu+1}(\kappa)}{\Lambda_{\nu-1}(\kappa)}\bar{a}_l, \quad \text{(A.8)}$$

where $g = \kappa^2$ and $\nu = l + 1/2$.

The coupling constant $g = g_{nl}$ corresponding to emergence of the nl level is determined from the equation $\Lambda_{\nu-1}(g^{1/2}) = 0$. We also have $N_l = (2l+1)/2(2l-1)$ and $J_l = (2l+1)/2l$. For the effective range $r_l^{(s)}$ at $g = g_{nl}$, we obtain formula (A.6) in which the coefficient $\tilde{\beta}_l$ is replaced with

$$\beta_l = \frac{2l+3}{2(2l+1)}\tilde{\beta}_l = (2l+1)!!(2l-3)!!, \quad R_l = r_s = R \quad \text{(A.9)}$$

($\beta_0 = -1$). Thus, $r_l^{(s)}$ and R_l are independent of the level number in this case,[6)] the quantity R_l being coincident with the well radius R for all n and l.

[5)] For the first time, Zeldovich [9], who used an example from solid-state physics, considered this effect for s-wave states. A generalization to the case of $l \geq 1$ was given in [10]. Possibly, the atomic spectrum with $l = 1$ undergoes rearrangement in a $K^-\alpha$ system [11].

[6)] Of course, this is a special feature of the model being considered.

(3) For the δ-function potential $v(x) = \delta(1 - x)$, a simple calculation yields $g_l = 2l + 1$, $N_l = 2(2l + 1)/(2l + 3)(2l - 1)$, $J_l = (2l + 1)/2l(l + 1)$, and

$$\beta_l = \frac{2}{2l+1}\tilde{\beta}_l, \quad R_l = \left(\frac{2l+3}{4}\right)^{\frac{1}{2l-1}}R \qquad (A.10)$$

(recall that, in this case, only one bound state appearing at $g = g_l$ can exist in each partial wave).

(4) Let us finally consider the potential (17), which reduces to a square-well potential at $\alpha = 0$ and to a cutoff Coulomb potential at $\alpha = 1$ (no. 4 in Table 2). The wave function corresponding to zero energy has the form

$$\chi_l(x) = \begin{cases} \text{const } x^{1/2} J_\mu\left(\kappa x^{\frac{2-\alpha}{2}}\right), & 0 < x < 1 \\ x^{-l}, & x > 1, \end{cases}$$

where

$$\kappa = 2g^{1/2}/(2-\alpha), \quad \mu = (2l+1)/(2-\alpha). \quad (A.11)$$

The scattering length

$$a_l^{(s)} = -\bar{a}_l \frac{J_{\mu+1}(\kappa)}{J_{\mu-1}(\kappa)} = \tilde{a}_l\left[1 - \frac{\Lambda_\mu(\kappa)}{\Lambda_{\mu-1}(\kappa)}\right] \qquad (A.12)$$

has poles at the points $\kappa_{nl} = \xi_n^{(\mu-1)}$, where $\xi_n^{(\nu)}$ is the nth positive zero of the Bessel function $J_\nu(\xi)$ corresponding to the coupling-constant value $g = g_{nl}$ at which the nl bound state appears. These coupling-constant values are given by

$$g_{nl} = \left[\left(1 - \frac{\alpha}{2}\right)\xi_n^{(\mu-1)}\right]^2, \quad n = 1, 2, \ldots. \qquad (A.13)$$

For $g = g_{nl}$ and $l \geq 1$, the wave function is normalizable; that is, we have

$$N_l = \frac{1}{2l-1} + \frac{2}{2-\alpha}\left[\kappa^{\frac{2}{2-\alpha}} J_\mu(\kappa)\right]^{-2} \int_0^\kappa J_\mu^2(y) y^{\frac{2+\alpha}{2-\alpha}} dy,$$

$$J_l = \frac{1}{2l} + \frac{2}{2-\alpha}\left[\kappa^{\frac{1}{2-\alpha}} J_\mu(\kappa)\right]^{-2} \int_0^\kappa J_\mu^2(y) y^{\frac{\alpha}{2-\alpha}} dy, \qquad (A.14)$$

where $\kappa = \kappa_{nl}$ and $J_{\mu-1}(\kappa) = 0$. These integrals are calculated analytically at $\alpha = 0$ and $\alpha = 1$ {see formulas (1.8.3.12) and (1.8.3.4) in [16]}. In the latter case, we obtain $\mu = 2l + 1$ and

$$r_l^{(s)} = -\frac{4}{3}(l+1)\left\{1 + \frac{2(4l^2-1)}{\left[\xi_n^{(2l)}\right]^2}\right\}\beta_l R^{1-2l}, \qquad (A.15)$$

where β_l is the coefficient given by (A.9). For s states, we have

$$r_s \equiv r_0^{(s)} = \frac{4}{3}\left\{1 - 2\left[\xi_n^{(0)}\right]^{-2}\right\}R, \qquad (A.16)$$

where $\xi_1^{(0)} = 2.405$, $\xi_2^{(0)} = 5.520$, and $\xi_n^{(0)} = \left(n - \frac{1}{4}\right)\pi + (8\pi n)^{-1} + \ldots$, $n \gg 1$ (see [17]). For $n \longrightarrow \infty$, r_s tends to $4R/3$—that is, to the effective-range value for the δ-function potential (however, this is not true for $l \neq 0$). For other α values, the integrals in (A.14) can easily be calculated numerically. The above analytic formulas at $\alpha = 0$ and $\alpha = 1$ were used as a check on the numerical calculations.

With increasing orbital angular momentum l, the effective ranges increase very fast in magnitude. For the square-well potential, formula (A.9) yields

$$\beta_1 = 3, \quad \beta_2 = 15, \quad \beta_3 = 315, \quad \beta_4 = 14\,175,$$
$$\beta_5 = 1091475, \quad \beta_6 = 127702575, \quad \beta_7 = 2.107(10),$$
$$\beta_{10} = 4.738(17), \quad \beta_{15} = 4.096(31), \ldots.$$

For $\alpha = 1$ and $n = 1$, the coefficients β_l take the values of

3.273, 18.252, 4.102(2), 1.935(4), 1.544(6), ...

for $l = 1, 2, 3, 4, 5$, respectively. In parentheses, we indicate exponents of the numbers—that is, $a(b) \equiv a \times 10^b$. For the δ-function potential, we have

$$\beta_1 = 12/5, \quad \beta_2 = 60/7, \quad \beta_3 = 140, \quad \beta_4 = 5.155(3),$$
$$\beta_5 = 3.358(5), \quad \beta_{10} = 8.240(16), \quad \beta_{15} = 4.965(30), \ldots.$$

In the limit $l \longrightarrow \infty$, we arrive at

$$\left|r_l^{(s)}\right| \approx \text{const}(2l)!(1/R^2)^l l^\sigma, \qquad (A.17)$$

where $\sigma = -1/2$ for the hard-sphere and square-well potentials, and $\sigma = -3/2$ for the δ-function potential. Thus, we have obtained the factorial asymptotic behavior. By using the of $1/n$ expansion [7], we can show that this is true for an arbitrary short-range potential.

APPENDIX 2

Let us briefly discuss the formulas for calculating Coulomb corrections. We consider the case of arbitrary l value, because there is no substantial simplification at $l = 1$.

In the asymptotic region, the wave function in the potential (15) for a zero-energy state has the form

$$\varphi_l(r, k^2 = 0) \approx r^{-l} - \frac{\alpha_l}{a_l^{(s)}} r^{l+1}, \quad r \longrightarrow \infty, \qquad (A.18)$$

where α_l is the numerical coefficient given by (A.7). The term $\frac{d^2}{dr^2}(r^{-l})$ in the Schrödinger equation must be much greater than $v(r)r^{l+1}$. This constrains the rate at which the potential can decrease [8]: if $v(r) \sim r^{-\beta}$ for $r \longrightarrow \infty$, we have $\beta > 2l + 3$.

At the exact resonance ($g = g_l$ and $a_l^{(s)} = \infty$), the function $\chi_l(r) \equiv \varphi_l(r, 0))$ satisfies the equation

$$\chi_l'' + \left[g_l v(r) - \frac{l(l+1)}{r^2}\right]\chi_l(r) = 0 \qquad (A.19)$$

and the conditions

$$\chi_l(0) = 0, \quad \chi_l(r) = r^{-l} + O(r^{-(l+\beta)}), \quad r \longrightarrow \infty. \qquad (A.20)$$

Setting

$$\varphi_l(r, k^2) = \varphi_l(r, 0) - \frac{k^2}{2}\vartheta_l(r) + O(k^4) \qquad (A.21)$$

for $E = k^2/2 \longrightarrow 0$, we find that, at $g = g_l$, the function $\varphi_l(r)$ introduced in (21) satisfies the equation

$$\vartheta_l'' + [g_l v(r) - l(l+1)r^{-2}]\vartheta_l = 2\chi_l(r) \qquad (A.22)$$

and the conditions

$$\vartheta_l(0) = 0, \quad \vartheta_l(r) = c_1 r^{l+1} + c_2 r^{\mu} + O(r^{\nu}), \quad r \longrightarrow \infty. \qquad (A.23)$$

Using Bethe's trick [14], we arrive at

$$\chi_l\vartheta_l' - \chi_l'\vartheta_l = 2\int_0^r \chi_l^2 dr = 2\left[N_l - \int_r^\infty \chi_l^2 dr\right].$$

Calculating the Wronskian with allowance for the asymptotic behavior of the functions $\chi_l(r)$ and $\vartheta_l(r)$ for $r \longrightarrow \infty$, we obtain

$$c_1 = \frac{2N_l}{2l+1}, \quad c_2 = -\frac{1}{2l-1}, \qquad (A.24)$$
$$\mu = 2 - l, \quad \nu = 2 - \beta - l,$$

where $l \geq 1$, and $\beta > 0$ is the exponent of decrease of the potential at infinity. We note that the constant N_l is related to the effective range $r_l^{(s)}$ corresponding to the coupling-constant value at which the relevant level appears [see equation (6)].

Equations (A.22)–(A.24) determine the function $\vartheta_l(r)$ unambiguously.[7)] At $l = 1$, the coefficients in the Coulomb corrections (12) are expressed in terms of the effective range $r_1^{(s)}$ and the Coulomb radius r_1^C of the system as

$$r_1^{(s)} = x_1^s / r_N, \quad r_1^C = x_1^C r_N, \qquad (A.25)$$

$$\ln x_1^C = \ln z + \int_0^\infty \left[\Theta(x-z) - \frac{2}{3}N_1 x + \chi_1(x)\vartheta_1(x)\right]\frac{dx}{x}, \qquad (A.26)$$

$$h_1 = k - \ln(x_1^s x_1^C), \qquad (A.27)$$

[7)] Indeed, the solution const$\chi_l(r) \propto r^{-l}$ to the homogeneous equation cannot be added to the function $\vartheta_l(r)$, because, in view of the inequality $-l > \nu$, this would change the asymptotic behavior in (A.23) (we assume that $\beta > 2$; that is, the potential $V_s(r)$ decreases at infinity faster than the centrifugal potential).

where $k = 1 - 2C + \ln(3/2) = 0.251$, $C = 0.577\ldots$ is the Euler constant, $x = r/r_N$, and z is an arbitrary parameter that does not affect the Coulomb radius. The Heaviside function $\Theta(x - z)$ ensures convergence of the integral in (A.26) at the upper limit. The above formulas make it possible to calculate the Coulomb corrections for an arbitrary local potential of the form (15). Numerical solutions to equations (A.19) and (A.22) for χ_1 and ϑ_1 can be constructed by means of a procedure that is similar to that used in [1] to treat the case of the s wave.

In order that the effective range $r_l^{(s)}$ exist, the condition $\lim_{r\to\infty} r^{2l+5}V(r) = 0$ or $\beta > 2l + 5$ must be satisfied (the scattering length is well defined under the less stringent condition $\beta > 2l + 3$). For potentials decreasing exponentially at infinity like the Yukawa potential, the scattering length and effective range are well defined for all l values.

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